이학박사 학위논문

Global Hypersurfaces of Section and the Spatial Kepler Problem

(대역적 절단 초곡면과 3차원 케플러 문제)

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Global Hypersurfaces of Section and the Spatial Kepler Problem

A dissertation
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Abstract

In the first part of this paper, we demonstrate the existence of global hypersurfaces of section of the Hamiltonian vector field of mechanical Hamiltonians of convex type, and the geodesic vector fields of convex hypersurfaces. The result can be regarded as a generalization of Birkhoff's annulus, and we provide many examples including the Kepler problem.

In the second part, we investigate the periodic orbits of rotating Kepler problem. We introduce a way to describe the moduli space of periodic orbits by angular momentum and Laplace-Runge-Lenz vector. We then classify every periodic orbit and compute the Conley-Zehnder index. The result can be interpreted by symplectic homology.

Key words: Global hypersurface of section, Kepler problem, symplectic geometry, Hamiltonian dynamics, Conley-Zehnder index
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Contents

Abstract **i**

1 Introduction **1**

2 Preliminaries **3**

2.1 Riemannian Geometry 3

2.1.1 Connections 3

2.1.2 Geodesics 6

2.1.3 Curvatures 11

2.2 Symplectic Geometry 15

2.2.1 Symplectic Manifolds 15

2.2.2 Contact Manifolds 18

2.2.3 Liouville Domains 19

2.3 Hamiltonian Dynamics 20

2.3.1 Hamiltonian Dynamics 20

2.3.2 Poisson Brackets 23

2.3.3 Hamiltonian Actions 26

2.3.4 Geodesic Flow as a Hamiltonian Flow 27

2.4 Global Hypersurfaces of Section 30

2.4.1 Global Hypersurfaces of Section 30

2.4.2 Open Book Decompositions 33

2.4.3 Global Hypersection of Sections of Reeb Flow 34

2.5 Floer Homology 36

2.5.1 Conley-Zehnder Index 36

2.5.2 Floer Homology 39
2.5.3 Symplectic Homology 41
2.5.4 S^1-equivariant Symplectic Homology 43
2.5.5 Morse-Bott Spectral Sequence 45

3 Existence of Global Hypersurfaces of Section 48
3.1 Mechanical Hamiltonian Flow 48
3.1.1 Mechanical Hamiltonian of Convex Type 49
3.1.2 Proof of the Existence 50
3.1.3 Extension of Return Map 53
3.1.4 Examples 53
3.2 Geodesic Flow on Convex Hypersurfaces 56
3.2.1 Geodesic Flows on Hypersurfaces 56
3.2.2 Locally Symmetric Convex Hypersurfaces 57
3.2.3 Proof of the Existence 61
3.2.4 Extension of Return Map 63
3.2.5 Examples 65

4 Periodic Orbits of Spatial Kepler Problem 73
4.1 Kepler Problem 73
4.1.1 Invariants of Kepler Problem 74
4.1.2 Three Laws of Kepler 76
4.1.3 Moser Regularization 80
4.2 Planar Rotating Kepler Problem 83
4.2.1 Moser Regularization 85
4.2.2 Periodic Orbits 87
4.3 Spatial Rotating Kepler Problem 91
4.3.1 Moser Regularization and Vertical Collision Orbits . . 92
4.3.2 Moduli Space of Spatial Kepler Orbits 93
4.3.3 Periodic Orbits 99
4.4 Conley-Zehnder Index of Periodic Orbits 102
4.4.1 Index of Retrograde and Direct Orbits 102
4.4.2 Index of Vertical Collision Orbits 110
4.4.3 Relation with Symplectic Homology 115

Bibliography **118**

Abstract (in Korean) **128**

Acknowledgement (in Korean) **129**

Chapter 1

Introduction

Symplectic geometry is a rich and vibrant field, with one of its central themes being the study of periodic Hamiltonian orbits and Reeb orbits. Two cornerstone conjectures highlight the importance of this topic: the celebrated Arnold conjecture [Arn04], which explores the relationship between the topology of a symplectic manifold and periodic Hamiltonian orbits, and the Weinstein conjecture [Wei79], which asserts the existence of periodic Reeb orbits. These foundational problems have spurred the development of numerous powerful tools and results, most notably Floer homology [Flo89].

This thesis is centered on two interconnected topics: the theory of global hypersurfaces of section and the periodic orbits of the rotating Kepler problem. Global hypersurfaces of section, originally introduced by Poincaré in his pioneering work on celestial mechanics [Poi87] and further developed by Birkhoff [Bir66], provide an elegant method for simplifying the study of dynamical systems. By reducing the dynamics of a vector field on an n-dimensional manifold to the dynamics of a diffeomorphism on an $(n-1)$-dimensional manifold, these hypersurfaces enable a profound simplification of the system's complexity. Ghys aptly referred to the existence of such structures as a "paradise for dynamicists" [Ghy09], highlighting their significance, particularly in the context of three-dimensional dynamics. A notable breakthrough in this area was achieved by Hofer, Wysocki, and Zehnder, who constructed disk-like global surfaces of section on dynamically convex

S^3 and proved the existence of exactly two or infinitely many periodic Reeb orbits [HWZ98]. Subsequent developments in this line of research can be found in [HSa11], [HMSa15], and [SaH18].

However, constructing global hypersurfaces of section in higher dimensions presents significant challenges, particularly due to the instability of the boundary, which must be a codimension-2 invariant submanifold of the dynamical system. To address this issue, Moreno and van Koert [MvK22b] proposed introducing symmetries into the system to ensure the existence of invariant submanifolds, providing a concrete example in the context of the restricted three-body problem. In this thesis, we establish the existence of global hypersurfaces of section for two distinct classes of dynamical systems. The first class comprises mechanical Hamiltonian systems of convex type, while the second involves geodesic flows on convex hypersurfaces in Euclidean spaces. The results pertaining to the second class are part of a joint work with Sunghae Cho [CL24] and include an analysis of the Kepler problem. We anticipate that these findings will extend to broader classes of examples and contribute to the development of higher-dimensional dynamical systems theory.

The second focus of this thesis is the Kepler problem, one of the most fascinating examples of Hamiltonian dynamical systems. The rotating Kepler problem, inspired by the restricted three-body problem, resolves the degeneracy of Keplerian orbits and provides a framework for computing the Conley–Zehnder index and applying Floer theory. Building on the foundational work of Albers, Fish, Frauenfelder, and van Koert [AFFvK13], which computed the Conley–Zehnder index for periodic orbits of the planar rotating Kepler problem, we extend these results to the spatial case. Specifically, we compute the Conley–Zehnder indices of isolated periodic orbits in the spatial rotating Kepler problem. Furthermore, by analyzing the moduli space of spatial Keplerian orbits, we employ the Morse–Bott spectral sequence to compute the Conley–Zehnder indices of degenerate orbits. These results are expected to have significant applications to the study of the restricted three-body problem, which can be understood as a perturbation of the Kepler problem.

Chapter 2

Preliminaries

This chapter introduces basic notions of Riemannian geometry, symplectic geometry and symplectic dynamics. We assume that every manifold, map, vector field and tensor is smooth for the rest of the paper. We refer [Spi79a] and [Lee13] for the general properties of smooth manifolds.

2.1 Riemannian Geometry

In this section, we introduce basic notions of Riemannian geometry including connections, geodesics, geodesic flows and curvatures. We refer [Mil63], [KN63], [KN69], [CE75] and [Spi79b] for the general reference for the Riemannian geometry.

2.1.1 Connections

Let M be a manifold. A symmetric positive definite 2-tensor g is called a **metric**, and the pair (M, g) is called a **Riemannian manifold**. This can be understood as an extension of the inner product on a tangent space to the entire tangent bundle.

Let (M, g) be a Riemannian manifold, and let $N \subset M$ be a submanifold. Then N is naturally a Riemannian manifold equipped with the restricted metric $g|_N$. If there exists a diffeomorphism $\varphi : (M, g_M) \to (N, g_N)$ such that $\varphi^* g_N = g_M$, we call φ an **isometry** and we say that (M, g_M) and (N, g_N) are **isometric**.

Example 2.1.1. The tangent bundle of Euclidean space can be described as $T\mathbb{R}^n = \left\{\sum p_i (\partial_{q_i})_q : q \in M\right\}$. The standard Euclidean metric on $\mathbb{R}^n$ is defined by $g = \sum dq_i \otimes dq_i$. A round $(n-1)$-sphere can be regarded as a submanifold defined by the level set of $f(q) = |q|^2$, equipped with the inherited metric.

Let M be a manifold. A **connection** is a bilinear operator ∇ defined on the space of vector fields on M, satisfying the following properties:

1. $\nabla_{fX}(Y) = f\nabla_X Y$,
2. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

where $X(f)$ is the directional derivative of f in the direction of X. A connection allows us to differentiate vector fields without using local coordinates. In this sense, ∇ is also called the **covariant derivative**. Once ∇ is defined on TM, we can extend ∇ to any tensor field on the tangent and cotangent bundles by applying the Leibniz rule and the pairing of vector fields with differential forms.

Note 2.1.2. A connection can be defined on any vector bundle or principal bundle. In general, a connection distinguishes the horizontal direction in the bundle, which is equivalent to choosing an embedding of the tangent bundle of the base manifold into the tangent bundle of the total space. More detailed descriptions of connections can be found in [KN63] or [Spi79b].

The choice of connection is not canonical in general, but there is a specific connection for Riemannian manifolds. A connection ∇ on a Riemannian manifold (M, g) is called the **Levi-Civita connection** if it satisfies the following conditions:

1. (Compatibility) $\nabla g = 0$, or equivalently,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

2. (Torsion-free) $\nabla_X Y - \nabla_Y X = [X, Y]$,

where $[X, Y]$ is the Lie bracket of X and Y. The following theorem can be found in many references, including [KN63], [Mil63], [Spi79b] or [dC92].

Proposition 2.1.3 (Fundamental Theorem of Riemannian Geometry)**.** *The Levi-Civita connection uniquely exists for any Riemannian manifold.*

Proof. Let X, Y, Z be vector fields. We have

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

It follows that

$$\begin{aligned} & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad = g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X) \\ &\quad = 2g(\nabla_X Y, Z) - g([X,Y], Z) + g([Y,Z], X) - g([Z,X], Y) \end{aligned}$$

We conclude that

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2} & \left(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \right. \\ & \left. + g([X,Y], Z) - g([Y,Z], X) + g([Z,X], Y) \right). \end{aligned}$$

Since every term on the right-hand side is independent of ∇, we have shown the existence and uniqueness of the Levi-Civita connection. □

For explicit computations, we assign a coordinate chart. Let $(e_1, \cdots, e_n)$ be a frame on a local chart of (M, g). For given ∇, we can write

$$\nabla_{e_i} e_j = \sum_k \Gamma_{ij}^k e_k,$$

and conversely Γ_{ij}^k characterizes ∇ on local chart. We call Γ_{ij}^k the **Christoffel symbols**. For the simplicity, we will use the following notations in the rest of this section while working with local coordinates:

1. We write $e^i = e_i^*$ for the dual frame, i.e., $e^i(e_j) = \delta^i_j$.
2. For a matrix $A = (a_{ij})$, we write $A^{-1} = a^{ij}$.
3. (Einstein convention) Repeated indices (one lower and one upper) imply summation. For example, $\sum_i X^i \partial_i = X^i \partial_i$.
4. Let ∂_i denote the i-th partial derivative in the given chart, and let f be a function. Then we write $\partial_i f = f_{,i}$.

Lemma 2.1.4. *Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection. Then Christoffel symbols satisfies following identities.*

1. $\Gamma^k_{ij} = \Gamma^k_{ji}$.
2. $\partial_i g_{jk} = \Gamma^a_{ij} g_{ak} + \Gamma^b_{ik} g_{jb}$.
3. $\Gamma^k_{ij} = \frac{1}{2} g^{ka} \left(g_{ai,j} + g_{ja,i} - g_{ij,a} \right)$

Proof. The first two properties follow directly from the defining conditions of the Levi-Civita connection. The third property is the local form of the formula derived in the proof of Proposition 2.1.3. □

2.1.2 Geodesics

There are many different approaches to defining the geodesics of a Riemannian manifold. We follow the definition from [Mil63] and [KN63].== We first define the notion of parallel transport. Let M be a Riemannian manifold, ∇ be the Levi-Civita connection and $\gamma : [a, b] \to M$ be a curve. Let V be a vector field along γ, meaning that V is a section of $\gamma^* TM$. With a local frame (e_i), we can write $V_t = V^i(t)(e_i)_{\gamma(t)}$. We denote $\dot{\gamma} = \dot{\gamma}^i e_i$ for the time derivative of γ. The **covariant derivative of V along γ** is defined by

$$\nabla_{\dot{\gamma}} V = \dot{\gamma}^j V^i_{,j} e_i + V^i \nabla_{\dot{\gamma}} e_i$$

Let $q \in M$ and $v \in T_p M$. Let $\gamma : [a, b] \to M$ be a curve in M such that $\gamma(a) = q$ and $\gamma(b) = q'$. Then, there exists a unique vector field V such that $V_p = v$ and $\nabla_{\dot{\gamma}} V = 0$, due to the existence and uniqueness of the solution to the differential equation. We call $v' = V_{q'}$ a **parallel transport** of v along γ. The curve γ is a **geodesic** if

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

In this sense, we say the geodesic is self-parallel.

Proposition 2.1.5 (Geodesic Equation)**.** *In local coordinate, a geodesic γ satisfies*

$$\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0.$$

In particular, the geodesic γ is determined by the initial conditions $\gamma(t_0)$ and $\dot{\gamma}(t_0)$.

Proof. This directly follows from the local formulation of parallel transport by substituting $\dot{\gamma}$ for V. The second statement follows from the existence and uniqueness of the solution to the second-order differential equation. □

Corollary 2.1.6. *Let γ be a geodesic. Then $|\dot{\gamma}|$ is constant along γ.*

Proof. We have

$$\begin{aligned}
\nabla_{\dot{\gamma}} g(\dot{\gamma}(t), \dot{\gamma}(t)) &= \nabla_{\dot{\gamma}} g_{ij}\dot{\gamma}^i\dot{\gamma}^j = (\nabla_{\dot{\gamma}} g_{ij})\dot{\gamma}^i\dot{\gamma}^j + 2g_{ij}(\nabla_{\dot{\gamma}}\gamma^i)\gamma^j \\
&= g_{ij,k}\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k - 2g_{ij}\Gamma^i_{kl}\dot{\gamma}^j\dot{\gamma}^k\dot{\gamma}^l \\
&= \left(\Gamma^a_{ij}g_{ak} + \Gamma^b_{ik}g_{jb}\right)\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k - 2g_{ij}\Gamma^i_{kl}\dot{\gamma}^j\dot{\gamma}^k\dot{\gamma}^l = 0
\end{aligned}$$

We used the geodesic equation Proposition 2.1.5 in the third equality and Lemma 2.1.4 in the fourth equality. □

Example 2.1.7. In Euclidean space $\mathbb{R}^n$, the metric $g_{ij} = \delta_{ij}$ is constant, so the Christoffel symbols vanish. Thus, the geodesic equation is $\ddot{\gamma}^i = 0$, which implies that the geodesics in Euclidean space are straight lines.

Example 2.1.8. Let $S^n \subset \mathbb{R}^{n+1}$ be a round n-sphere. It's well-known that geodesics on S^n are great circles. To see this, we need to use a local coordinate on S^n. We use **stereographic projection**, which is given by the formula

$$\varphi : S^n \setminus \{(1,0,\cdots,0)\} \to \mathbb{R}^n$$
$$(x_0, x_1, \cdots, x_n) \mapsto \left(\frac{x_1}{1-x_0}, \cdots, \frac{x_n}{1-x_0}\right).$$

The inverse is given by

$$\varphi^{-1} : \mathbb{R}^n \to S^n \setminus \{(1,0,\cdots,0)\}$$
$$(q_1, \cdots, q_n) \mapsto \left(\frac{1-|q|^2}{1+|q|^2}, \frac{2q_1}{1+|q|^2}, \cdots, \frac{2q_n}{1+|q|^2}\right).$$

We have

$$\partial_{q_i} = \frac{-4q_i}{(1+|q|^2)^2}\partial_{x_0} - \sum_{k=1}^{n} \frac{4q_iq_k}{(1+|q|^2)^2}\partial_{x_k} + \frac{2}{1+|q|^2}\partial_{x_i}$$

and it follows

$$\begin{aligned} g_{ij} = g\left(\partial_{q_i}, \partial_{q_j}\right) &= \frac{16q_iq_j}{(1+|q|^2)^4} + \frac{16q_iq_j(1+|q|^2)}{(1+|q|^2)^4} \\ &\quad - \frac{16q_iq_j}{(1+|q|^2)^3} + \frac{4}{(1+|q|^2)^2}\delta_{ij} \\ &= \frac{4}{(1+|q|^2)^2}\delta_{ij} \end{aligned}$$

Using the formula in Lemma 2.1.4, we obtain

$$\Gamma^i_{ii} = -\frac{2q_i}{1+|q|^2}, \quad \Gamma^k_{ik} = -\frac{2q_i}{1+|q|^2}, \quad \Gamma^i_{kk} = \frac{2q_i}{1+|q|^2}$$

for $i \neq k$, and otherwise $\Gamma^i_{jk} = 0$. Substituting into the geodesic equation, we have

$$\ddot{\gamma}^i - \sum_{k=1}^{n} \frac{2q_k}{1+|q|^2}\dot{\gamma}^i\dot{\gamma}^k + \sum_{k\neq i} \frac{2q_i}{1+|q|^2}\dot{\gamma}^k\dot{\gamma}^k = 0$$

Now let γ be a great circle lies on q_0q_1-plane starting at the south pole:

$$\gamma(t) = (-\cos t, \sin t, 0, \cdots, 0) \in S^n \subset \mathbb{R}^{n+1}.$$

Under the stereographic projection, we have

$$\gamma(t) = \left(\frac{\sin t}{1+\cos t}, 0, \cdots, 0\right).$$

Then, we have

$$\dot{\gamma}^1 = \frac{1}{1+\cos t}, \quad \ddot{\gamma}^1 = \frac{\sin t}{(1+\cos t)^2}$$

For $i \neq 1$, every term in the geodesic equation is zero. If $i = 1$, we have

$$\ddot{\gamma}^1 - \frac{2q_1}{1+|q|^2}(\dot{\gamma}^1)^2 = \frac{\sin t}{(1+\cos t)^2} - \frac{2\sin t}{1+\cos t}\frac{(1+\cos t)^2}{2(1+\cos t)}\frac{1}{(1+\cos t)^2} = 0.$$

Thus, γ is a geodesic. Any other great circle can be mapped to γ by an $SO(n+1)$-action on S^n, so every great circle is a geodesic. Conversely, for any initial condition of a geodesic, we can find a great circle with the same initial condition. This proves that every geodesic is a great circle.

Let $\gamma : [a, b] \to M$ be a curve. The **length functional** is a function $L : C^\infty([a, b], M) \to \mathbb{R}$ defined by

$$L(\gamma) = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt$$

Here's a theorem, which was chosen as the definition of geodesic in [CE75] and [Spi79b].

Proposition 2.1.9. *A curve $\gamma : [a, b] \to M$ is a geodesic if and only if it is a critical point of the length functional L.*

Proof. See [Mil63], Chapter 12. □

A Riemannian manifold (M, g) can be regarded as a metric space by introducing the distance

$$d_g(q_0, q_1) = \inf_{\substack{\gamma(t_0)=q_0 \\ \gamma(t_1)=q_1}} L(\gamma).$$

This infimum should be realized by a minimizing geodesic from q_0 to q_1, if one exists.

Theorem 2.1.10 (Hopf-Rinow)**.** *Let (M, g) be a Riemannian manifold. Then the following statements are equivalent.*

1. *A closed and bounded subset of M is compact.*
2. *The manifold M is complete as a metric space.*
3. *The manifold M is **geodesically complete**, meaning that any geodesic can be extended infinitely in both directions.*

Proof. See [KN63], Chapter IV. □

The **exponential map** is defined by

$$\begin{aligned} \exp : TM &\to TM \\ (q, p) &\mapsto (\gamma(1), \dot{\gamma}(1)), \end{aligned}$$

where γ is the geodesic starting at q with initial velocity p. By Theorem 2.1.10, the exponential map is globally defined if M is a complete manifold. We can define a time-dependent diffeomorphism on TM by

$$\varphi_t(q, p) = \exp_q(tp),$$

which is called the **geodesic flow** of M. Notice that φ_0 is the identity.

If a time-dependent diffeomorphism φ_t from a manifold to itself is given and $\varphi_0 = \mathrm{Id}$, we can find a vector field X such that $\varphi_t = Fl_t^X$, the flow of X, by differentiating with respect to t at $t = 0$. It follows that for a Riemannian manifold (M, g), there exists a unique vector field X_g which generates the geodesic flow, and it is called the **geodesic vector field**.

Let S_rTM denote the subset of TM consists of vectors of length $r \geq 0$. Since the geodesic flow preserves the length by Corollary 2.1.6, we can see that S_rTM is invariant subset of X_g. In particular, $M \simeq S_0TM$ is the zero locus of X_g.

Let $N \subset M$ be a submanifold of a Riemannian manifold (M, g). We call N a **totally geodesic submanifold** if exp maps TN into TN. By definition, a submanifold is totally geodesic if and only if it is invariant under the geodesic flow.

We call $i : M \to M$ an **isometric involution** if i is an isometry and $i^2 = \mathrm{Id}_M$. The isometric involution is very useful for finding totally geodesic submanifolds, as we can see in the following theorem.

Theorem 2.1.11. *Let (M, g) be a Riemannian manifold and N be a closed submanifold. Assume that there exist a tubular neighborhood $\nu(N)$ of N and that N is the fixed point locus of i. Then N is a totally geodesic submanifold.*

Proof. See [dC92], Chapter VII.8. □

Example 2.1.12. Recall Example 2.1.8. Consider the isometric involution defined on the sphere,

$$i : S^n \to S^n \subset \mathbb{R}^{n+1}$$
$$(x_0, x_1, \cdots, x_n) \mapsto (-x_0, x_1, \cdots, x_n).$$

The invariant subset of i is the equator

$$S^{n-1} = \left\{(0, x_1, \cdots, x_n) : \sum x_j^2 = 1\right\} \subset S^n \subset \mathbb{R}^{n+1}.$$

Since a great circle lies on the surface spanned by its initial position and velocity, S^{n-1} is a totally geodesic submanifold.

2.1.3 Curvatures

Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection. We define the **Riemann curvature tensor** as the following $(3,1)$-tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We present the local formula for the Riemann curvature tensor.

Lemma 2.1.13. *Let (M, g) be a Riemannian manifold and e_i be a local frame. We write $R(e_i, e_j)e_k = R^l_{ijk} e_l$, then the components are given by*

$$R^l_{ijk} = \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}$$

Let $q \in M$ and $v, w \in T_qM$. Denote $||v \wedge w||$ as the area of parallelogram spanned by v and w. We define the **sectional curvature** by

$$K_M(v,w) = \frac{g\left(R(v,w)w, v\right)}{||v \wedge w||^2}$$

Note that this definition only depends on the plane spanned by v and w, so K_M is defined on the 2-Grassmannian bundle of TM.

There are many qualitative explanation about what curvature measures. We introduce a formula whose derivation can be found in [CE75], Chapter 1.4. Let $\gamma : I \to M$ be a geodesic with $\gamma(0) = q$. Let α be a variation of γ, which means that $\alpha : I \times (-\varepsilon, \varepsilon) \to M$ and $\alpha(-, s) = \alpha_s(-) = \gamma$. For simplicity, we denote $\dot{\gamma} = T$, $\partial_s \alpha(-, s) = V$. Let $w \in T_pM$ and W be parallel extension of w along γ, which means $\nabla_{\dot{\gamma}} W = 0$ and $W_q = w$. Then we have the following Taylor expansion

$$||d\exp(tW)||^2 = t^2 - \frac{1}{3} g\left(R(w,T)T, w\right) t^4 + O(t^5).$$

If we further assume that w and T are orthonormal, we have

$$||d\exp(tW)||^2 = t^2 - \frac{1}{3} K_M(w,T) t^4 + O(t^5)$$

The left-hand side term $d\exp(tW)$ measures the deviation of geodesics. One can see that the geodesics get closer if K_M is positive, and scatter if K_M is negative. This property can also be observed in the Gaussian curvature of surfaces, see for example [dC76].

Example 2.1.14. As mentioned in Example 2.1.7, the Christoffel symbols of Euclidean space vanish, and so does the Riemann curvature. Hence the Euclidean space has constant sectional curvature 0.

Example 2.1.15. Let S^n be a round sphere. Using the result from Example 2.1.8 we can compute the sectional curvature of S^n in terms of the stereographic projection. We have

$$\begin{aligned}
\Gamma^i_{kk,i} &= \frac{\partial}{\partial q_i}\frac{2q_i}{1+|q|^2} = \frac{2}{1+|q|^2} - \frac{4q_i^2}{(1+|q|^2)^2},\\
\Gamma^i_{ik,k} &= \frac{\partial}{\partial q_k}\left(-\frac{2q_k}{1+|q|^2}\right) = -\frac{2}{1+|q|^2} + \frac{4q_k^2}{(1+|q|^2)^2},\\
\Gamma^p_{kk}\Gamma^i_{ip} &= \Gamma^i_{kk}\Gamma^i_{ii} + \Gamma^k_{kk}\Gamma^i_{ik} + \sum_{j\neq i,k}\Gamma^j_{kk}\Gamma^i_{ij} = \frac{4(q_i^2-q_k^2)}{(1+|q|^2)^2} - \sum_{j\neq i,k}\frac{4q_j^2}{(1+|q|^2)^2},\\
\Gamma^p_{ik}\Gamma^i_{kp} &= \Gamma^i_{ik}\Gamma^i_{ki} + \Gamma^k_{ik}\Gamma^i_{kk} = \frac{4(q_k^2-q_i^2)}{(1+|q|^2)^2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
R^i_{ikk} &= \Gamma^i_{kk,i} - \Gamma^i_{ik,k} + \Gamma^p_{kk}\Gamma^i_{ip} - \Gamma^p_{ik}\Gamma^i_{kp}\\
&= \frac{4}{1+|q|^2} - \frac{4|q|^2}{(1+|q|^2)^2} = \frac{4}{(1+|q|^2)^2}.
\end{aligned}$$

Therefore, the sectional curvature is

$$K_{S^n}(e_i,e_k) = \frac{g(R(e_i,e_k)e_k,e_i)}{||e_i\wedge e_k||^2} = \frac{g_{ii}R^i_{ikk}}{g_{ii}g_{kk}} = \frac{R^i_{ikk}}{g_{kk}} = 1.$$

This shows that S^n has constant sectional curvature 1, which is consistent with the fact that S^2 has constant Gaussian curvature 1.

Let $N \subset M$ be an oriented submanifold of codimension 1, and ν be the unit normal vector field on N. We define the **second fundamental form** S as an operator that assigns to each tangent vector of N a tangent vector of M by

$$S(v) = \nabla_v \nu.$$

This definition is consistent with the second fundamental form used in the geometry of surfaces, which can be found in [dC76].

Proposition 2.1.16. *Let $N \subset M$ be an oriented submanifold of codimen-*

sion 1. Then the following formula holds.

$$K_N(v,w) = K_M(v,w) + \frac{g(S(v),v)g(S(w),w) - g(S(v),w)^2}{g(v,v)g(w,w) - g(v,w)^2}$$

Proof. See [dC92], Theorem 6.2.5. □

The formula becomes simpler in the case of a hypersurface in Euclidean space, where the sectional curvature vanishes.

Corollary 2.1.17. *Let $M \subset \mathbb{R}^n$ be an oriented submanifold of codimension 1. Let v, w be orthonormal vector fields on TM. Then the following holds.*

$$K_M(v,w) = g(S(v),v)g(S(w),w) - g(S(v),w)^2.$$

Example 2.1.18. Consider the round sphere $S^n \subset \mathbb{R}^{n+1}$ mentioned in Example 2.1.8. We can take $\nu = \sum_{i=0}^n q_i \partial_{q_i}$ as a unit normal vector. For $v_i = \partial_{q_i}$, one can see that $S(v_i) = v_i$, and thus from Proposition 2.1.16, we have $K_N(v_i, v_j) = 1$. This result is consistent with Example 2.1.15.

We now focus on a specific type of manifolds: convex hypersurfaces in Euclidean space. We say a domain $D \subset \mathbb{R}^{n+1}$ is **convex** if for any $q_0, q_1 \in D$, the straight line segment connecting q_0 and q_1 lies in D. For example, the standard disc is convex. We call $M \subset \mathbb{R}^{n+1}$ a **convex hypersurface** if M is a regular level set of some function f and bounds a compact convex domain. For example, a standard round sphere is a convex hypersurface.

A function on $\mathbb{R}^{n+1}$ is **convex** if for any $q_0, q_1 \in \mathbb{R}^{n+1}$ and $0 \leq t \leq 1$,

$$f(tq_0 + (1-t)q_1) \leq tf(q_0) + (1-t)f(q_1).$$

If the strict inequality holds for $0 < t < 1$, we call f is **strictly convex**.

Lemma 2.1.19. *Let $M = f^{-1}(c) \subset \mathbb{R}^{n+1}$ be a regular level set.*

1. *If* $\mathrm{Hess}(f)$ *is positive definite on M, then f is strictly convex.*

2. *If f is strictly convex, then M is a convex hypersurface.*

3. *If M is a convex hypersurface, then M is diffeomorphic to a sphere.*

Proof. We only prove the third statement. Let M be convex, bounding a compact convex region D. After translation, we can assume that 0 is in the interior of D. We define $\varphi : M \to S^n$ by $\varphi(x) = x/||x||$. By convexity, φ is a smooth injective map, which must be a diffeomorphism. □

Remark 2.1.20. We can treat the case of f with a negative definite Hessian similarly, because in that case, we can use $-f$ instead of f. If $M = f^{-1}(c)$, then $M = (-f)^{-1}(-c)$, and with Lemma 2.1.19, we can see that M is still a convex hypersurface. In short, it's enough to require the Hessian of f to be either positive or negative definite to ensure that its regular level set is convex.

2.2 Symplectic Geometry

In this section, we introduce general notions of symplectic geometry, including symplectic manifolds, contact manifolds and Liouville domains. We refer to [CdS01], [Ber01] and [MS17] as general references for symplectic geometry.

2.2.1 Symplectic Manifolds

Let W be a manifold without boundary. A nondegenerate 2-form ω on W is called a **symplectic form** and we call (W, ω) a **symplectic manifold**. Note that a symplectic manifold must be even-dimensional, because every 2-form is degenerate in odd dimensions. Let (W_1, ω_1) and (W_2, ω_2) be symplectic manifolds and let $\varphi : W_1 \to W_2$ be a diffeomorphism. If $\varphi^*\omega_2 = \omega_1$, we say φ is a **symplectomorphism**. Consider the symplectomorphism $\varphi : (W, \omega) \to (W, \omega)$ where the symplectic form ω is exact, so $\omega = d\lambda$ for some 1-form λ. We call φ is an **exact symplectomorphism** if $\varphi^*\lambda - \lambda = df$ for some function f. Let $V \subset W$ be a submanifold such that $\omega|_V$ is also a symplectic form on V. We call V a **symplectic submanifold**.

Example 2.2.1. Consider $\mathbb{R}^{2n} = \{(q, p) \,:\, q, p \in \mathbb{R}^n\}$. Define 2-form

$$\omega = \sum dp_i \wedge dq_i.$$

Then it is clear that ω is a symplectic form. Here, $\mathbb{R}^{2n}$ can be regarded as a cotangent bundle of $\mathbb{R}^n$, and will be generalized in the next example.

Example 2.2.2. Let M be a manifold and $W = T^*M$ be its cotangent bundle. Let $(q,p) \in U$ be a local coordinate chart of W. To be precise, p_i represents the coefficient of dq_i in U. We define a **canonical 1-form** on W locally by

$$\lambda_{(q,p)} = pdq = \sum p_i dq_i$$

Using the coordinate changing formula, we can see that λ can be patched together to form a global 1-form. It's clear that the derivative $d\lambda = dp \wedge dq$ is a symplectic form.

We note that a diffeomorphism between manifolds induces a symplectomorphism between their cotangent bundles via pullback. Additionally, if N is a submanifold of M, then T^*N is a symplectic submanifold of T^*M.

Note 2.2.3. An important opposite concept to symplectic submanifold is **Lagrangian submanifold**, which is a submanifold N of dimension $\dim M/2$ such that $\omega|_N = 0$. This concept is crucial for defining Lagrangian Floer homology and the Fukaya category. See [CdS01] or [MS17] for basic and general discussions on Lagrangian submanifold.

Theorem 2.2.4 (Darboux Lemma)**.** *Let (W, ω) be a symplectic manifold. For any point in W, there exists a local chart U around the point with coordinates (q,p) such that*

$$\omega = \sum dp_i \wedge dq_i.$$

Proof. See [Arn89] Section 43.B or [CdS01] Theorem 8.1. □

The theorem means that any symplectic manifold is locally symplectomorphic to an open subset or $(\mathbb{R}^{2n}, \omega_{\text{std}})$. This implies that the local geometry of a symplectic manifold can be fully understood in terms of Euclidean space, which is distinct from that of Riemannian geometry.

We note the relation between symplectic forms and Riemannian metrics. Let W be an even-dimensional manifold. An **almost complex structure** on M is a bundle map $J : TW \to TW$ such that $J^2 = -\mathrm{Id}$. A complex structure on a complex manifold is an almost complex structure, but the converse is not generally true.

Let ω be a symplectic form on W and J be an almost complex structure on M. If

$$g(-,-) = \omega(-, J-)$$

defines a Riemannian metric, we call ω and J are **compatible** and (ω, J, g) a **compatible triple**.

Proposition 2.2.5. *Let W be an even-dimensional manifold. If any two of the following three are given,*

1. *A symplectic form ω*
2. *A Riemannian metric g*
3. *An almost complex structure J*

then there exists the third one which is compatible with the given two structures. In particular, any symplectic manifold admits a compatible almost complex structure.

Proof. See [CdS01] Chapter 12, 13. The last statement follows from the fact that any manifold admits a Riemannian metric. □

Example 2.2.6. Consider $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ equipped with the standard symplectic form $\omega = dp \wedge dq$. Let us denote $\partial_{p_i} = v_i$, $\partial_{q_i} = w_i$. Let J be a standard complex structure on $\mathbb{R}^{2n}$ such that $J(v_i) = w_i$, $J(w_i) = -v_i$. Then we can see that $\omega(-, J-)$ defines a standard Euclidean metric on $\mathbb{R}^{2n}$. With this basis, we can write

$$\omega(v, Jw) = v^t \Omega J w = v^t w$$

so that Ω, the matrix describes the symplectic form, is equal to $-J$. Under the basis $\{v_1, w_1, \cdots, v_n, w_n\}$, we have

$$\Omega = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

2.2.2 Contact Manifolds

Let Y be a $(2n+1)$-dimensional manifold, and ξ let be a distribution of codimension 1 on Y. We can locally write ξ as $\ker\alpha$ for some 1-form α. Assume that ξ is coorientable, which means that TW/ξ is orientable. In this case, we can find a globally defined 1-form α. If $\alpha \wedge (d\alpha)^n$ is a volume form, we say α is a **contact form**, $\xi = \ker\alpha$ is a **contact structure**, and (W, ξ) is a **contact manifold**.

Example 2.2.7. Let M be a Riemannian manifold, and let $Y = ST^*M$ be a unit cotangent bundle of M. Then the canonical 1-form defined in Example 2.2.2 is a contact form.

Let (Y_1, ξ_1) and (Y_2, ξ_2) be contact manifolds, and let $\varphi : Y_1 \to Y_2$ be a diffeomorphism. If $d\varphi\xi_1 = \xi_2$, we call (Y_1, ξ_1) and (Y_2, ξ_2) are **contactomorphic**, and φ is a **contactomorphism**. This can be understood as a diffeomorphism that preserves contact structure. Note that this condition does not require φ to preserve contact forms.

Similar to the symplectic manifold, the local geometry of contact manifold is trivial in the sense of the following theorem.

Theorem 2.2.8 (Darboux Lemma)**.** *Let $(Y, \ker\alpha)$ be a contact manifold. Then, for any point in Y, there exists a local chart U with coordinate (x, y, z) such that*

$$\alpha = \sum x_i dy_i + dz.$$

Proof. See [Arn89] Appendix 4. □

Let $(Y, \ker\alpha)$ be a contact manifold. A vector field R such that $\alpha(R) = 1$ and $i_R d\alpha = 0$ is called the **Reeb vector field**. From the definition, the Reeb vector field exists uniquely for a given contact form α and never vanishes. Note that the definition of the Reeb vector field depends on the choice of the contact form, so contactomorphic contact manifolds can admit different Reeb vector fields. The role of the Reeb vector field in this paper will be discussed in Section 2.3.1.

2.2.3 Liouville Domains

Let W be a manifold with boundary, and let $\omega = d\lambda$ be an exact 2-form such that $(\mathring{W}, \omega)$ is a symplectic manifold, where $\mathring{W}$ is the interior of W. A vector field X such that $i_X\omega = \omega(X, -) = \lambda$ is called a **Liouville vector field**. If a Liouville vector field X exists and defines and points outward on the boundary, we call (W, λ) a **Liouville domain**.

Example 2.2.9. Let M be a Riemannian manifold, and let $T^*_{\leq 1}M$ be the submanifold of T^*M consisting of cotangent vectors of length less or equal to 1. Consider the outward normal vector field X, locally defined by

$$X = \sum p_i \frac{\partial}{\partial p_i}.$$

Then for $\lambda = pdq$ and $\omega = dp \wedge dq$, we can see that $i_X\omega = \lambda$, which shows that $(T^*_{\leq 1}M, \lambda)$ is a Liouville domain.

The previous example implies that the boundary of Liouville domain can be regarded similarly to the unit cotangent bundle. Indeed, a Liouville domain can be completed to a symplectic manifold in a canonical way by attaching $\partial W \times [1, \infty)$ with an appropriate symplectic form, which is called the symplectization of ∂W, along its boundary. This notion will be used to define symplectic homology in Section 2.5.

Here is a connection between the Liouville domain and contact manifold, which is the main motivation for introducing the notion of Liouville domains.

Proposition 2.2.10. *Let (W, λ) be a Liouville domain and X be a Liouville vector field. Then, a hypersurface Y transverse to X is a contact manifold with contact form $\lambda|_Y$. In particular, $(\partial W, \ker \lambda|_{\partial W})$ is a contact manifold.*

Proof. Let $q \in Y$, and let $\{v_1, \cdots, v_{2n-1}\}$ be a basis of T_qY. Since Y is transverse to X, the set $\{X_q, v_1, \cdots, v_{2n-1}\}$ forms a basis of T_pW. Now we have

$$\begin{aligned} \lambda \wedge (d\lambda)^{n-1}(v_1, \cdots, v_{2n-1}) &= i_X\omega \wedge (d\lambda)^{n-1}(v_1, \cdots, v_{2n-1}) \\ &= \omega^n(X_q, v_1, \cdots, v_{2n-1}). \end{aligned}$$

By the nondegeneracy of ω, the result follows. □

Example 2.2.11. We can apply Proposition 2.2.10 to Example 2.2.9. It follows that the unit cotangent bundle ST^*M of a manifold is a contact manifold. A contact form is given by $\lambda = pdq$.

2.3 Hamiltonian Dynamics

In this section, we introduce basic notions of Hamiltonian dynamics including Hamiltonian vector fields, Poisson brackets and Hamiltonian action. We also describe the geodesic vector field as a Hamiltonian vector field. We refer to [HZ11] and [MS17] for general reference of Hamiltonian dynamics.

2.3.1 Hamiltonian Dynamics

Let (W, ω) be a symplectic manifold. Given a function $H : W \to \mathbb{R}$, we can associate a vector field X_H to H via

$$(i_{X_H})\,\omega(-) = \omega(X_H, -) = -dH(-).$$

The vector field X_H uniquely exists due to the non-degeneracy of ω. We say X_H is a **Hamiltonian vector field**, and in this sense, we call H a **Hamiltonian function** or simply **Hamiltonian**. We also say (M, ω, H) is a **Hamiltonian system**.

If a diffeomorphism from W to itself can be written as a time 1-flow of a Hamiltonian vector field, we say it is a **Hamiltonian diffeomorphism**. We might also use a time-dependent Hamiltonian $H : W \times \mathbb{R} \to \mathbb{R}$ to define time-dependent Hamiltonian diffeomorphism. The name "Hamiltonian" came from the Hamiltonian formulation in classical mechanics, which is explained in the following example.

Example 2.3.1. Let $W = T^*\mathbb{R}^n$, $\omega = \omega_{\text{std}} = dp \wedge dq$, and $H = H(q,p)$ be any function. Let the Hamiltonian vector field of H have the form $X_H =$

$\sum a_i\partial_{q_i} + b_i\partial_{p_i}$. By definition, we have

$$-i_X\omega = \sum (a_i dp_i - b_i dq_i) = \sum \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)$$

It follows that

$$X_H = \sum \left(\frac{\partial H}{\partial p_i}\partial_{q_i} - \frac{\partial H}{\partial q_i}\partial_{p_i} \right).$$

Let V be a function defined on the base manifold $\mathbb{R}^n$, and define a Hamiltonian on $T^*\mathbb{R}^n$ by

$$H(q,p) = \frac{1}{2}||p||^2 + V(q).$$

This is called a mechanical Hamiltonian, which describes the mechanical energy of a particle with a potential function V depending only on the position. From the above formula, we have

$$X_H = \sum \left(p_i\partial_{q_i} - \frac{\partial V}{\partial q_i}\partial_{p_i} \right).$$

The differential equation describing integral curve $\gamma = (x,y)$ of X_H can be written as

$$\dot{q}_i(t) = p_i(t), \qquad \dot{p}_i(t) = -\frac{\partial V}{\partial q_i}(t)$$

or equivalently

$$\ddot{p}(t) = -\frac{\partial V}{\partial q}(t)$$

This is exactly the same as the Newton's second law of motion.

In particular, let $V(q) = 0$, which describes a *free particle* without any external force. Then we can see that the equation of motion is $\ddot{q}(t) = 0$. This can be generalized to the geodesic on a Riemannian manifold, which is explained in Section 2.3.4.

Lemma 2.3.2. *A Hamiltonian diffeomorphism is a symplectomorphism.*

Proof. Let φ_t be a Hamiltonian diffeomorphism of a symplectic manifold (W,ω), generated by a Hamiltonian H. Then we have

$$\left(Fl^{X_H}\right)^*\omega = \mathcal{L}_{X_H}\omega = -di_{X_H}\omega = -d(dH) = 0.$$

We used Cartan's magic formula for the second equality. □

The key property of a Hamiltonian vector field X_H to define a symplectomorphism is that $di_{X_H}\omega = 0$. In this sense, we say a vector field X on a symplectic manifold is a **symplectic vector field** if $di_X\omega = 0$. The flow of a symplectic vector field is a symplectomorphism. Note that if $H^1(W) = 0$, then every closed form is exact, so every symplectic vector field is a Hamiltonian vector field.

Lemma 2.3.3. *Let H be a Hamiltonian defined on a symplectic manifold (W, ω), and X_H be a Hamiltonian vector field.*

1. $(X_H)_p = 0$ *if and only if* $dH_p = 0$.
2. X_H *is tangent to the regular level set of H. In other words, H is preserved under the Hamiltonian flow Fl^{X_H}.*

Proof. The first statement is trivial from the definition of a Hamiltonian vector field. For the second statement, we have $dH(X_H) = -\omega(X_H, X_H) = 0$. Note that the second statement can be understood as the conservation of mechanical energy. □

From Lemma 2.3.3, we can see that the Hamiltonian vector field preserves the level set of the Hamiltonian. Hence we can consider the Hamiltonian dynamics restricted to the level set. Let (W, ω) be a symplectic manifold and $H : W \to \mathbb{R}$ be a Hamiltonian. For a regular value c of H, we call $Y = H^{-1}(c)$ is **of contact type** if there exists a vector field X, which is positively transverse to Y, which means $dH(X) > 0$, and $\mathcal{L}_X\omega = \omega$. In this case, the sublevel set $W_c = H^{-1}(-\infty, c]$ is a Liouville domain with the Liouville vector field X, and from Proposition 2.2.10, we can see that $Y = \partial W_c$ is a contact manifold with the contact form $i_X\omega|_{Y}$,.

Proposition 2.3.4. *Let H be a Hamiltonian defined on a symplectic manifold (W, ω) and c be a regular value of H. If $Y = H^{-1}(c)$ is a level set of contact type, the Hamiltonian vector field X_H on $H^{-1}(c)$ is parallel to the Reeb vector field. Therefore, the Hamiltonian orbit of H is a reparametrization of the Reeb orbit.*

Proof. Let X be a Liouville vector field. We have

$$\begin{aligned} i_{X_H}\omega &= -dH = 0 \text{ on } H^{-1}(c) \\ i_{X_H} i_X \omega &= \omega(X, X_H) = dH(X) > 0. \end{aligned}$$

Since $X_H \neq 0$, X_H is parallel to the Reeb vector field. □

Corollary 2.3.5. *Let $H_1, H_2 : (M, \omega) \to \mathbb{R}$ be Hamiltonians and c_1, c_2 be regular values of H_1, H_2 such that $H_1^{-1}(c) = H_2^{-1}(c) = Y$. Then X_{H_1} and X_{H_2} are parallel on Y. In other words, the level set determines the Hamiltonian flow up to reparametrization.*

2.3.2 Poisson Brackets

Let $H, F : (W, \omega) \to \mathbb{R}$ be Hamiltonians. The **Poisson bracket** is defined by

$$\{H, F\} := \omega(X_H, X_F).$$

Lemma 2.3.6. *Let H, F be Hamiltonians on (W, ω). Then the followings hold:*

1. $\{H, F\} = dH(X_F) = X_F(H)$.
2. $\{H, F\} = -\{F, H\}$.
3. *If $(W, \omega) = (T^*\mathbb{R}^n, \omega_{\text{std}})$, we have*

$$\{H, F\} = \sum \frac{\partial H}{\partial x_j}\frac{\partial F}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial F}{\partial x_j}.$$

Proof. The first two statements can be directly derived from the definition. For the last statement, we use the formula given in Example 2.3.1. Using first statement of this lemma, we have

$$\begin{aligned} \{H, F\} = dH(X_F) &= \left(\sum \frac{\partial H}{\partial x_j} dx_j + \frac{\partial H}{\partial y_j} dy_j\right)\left(\sum \frac{\partial F}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial F}{\partial x_j}\frac{\partial}{\partial y_j}\right) \\ &= \sum \left(\frac{\partial H}{\partial x_j}\frac{\partial F}{\partial y_j} - \frac{\partial H}{\partial y_j}\frac{\partial F}{\partial x_j}\right). \end{aligned}$$

□

Lemma 2.3.6 implies that the Poisson bracket measures how much two Hamiltonian flows do not commute. In particular, the first statement shows that if $\{H, F\} = 0$, then H is preserved under the Hamiltonian flow of F and vice versa. Intuitively, in this case we can say that H and F are independent. In particular, if $\{H, F\} = 0$ we have

$$Fl_t^{X_{H+F}} = Fl_t^{X_H} \circ Fl_t^{X_F}$$

since their flow commutes. We note a celebrated theorem of Arnold and Liouville, the proof of which can be found in [Arn89].

Theorem 2.3.7 (Arnold-Liouville)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold and $F_i : M \to \mathbb{R}$, $i = 1, \cdots, n$ be Hamiltonians. Let $F = (F_1, \cdots, F_n) : M \to \mathbb{R}^n$, and let $c \in \mathbb{R}^n$ be a regular value of F. Assume that*

1. *The Hamiltonians F_i Poisson-commute, which means $\{F_i, F_j\} = 0$ for every i, j.*
2. *The regular level set $L = F^{-1}(c)$ is connected and compact.*

Then there exists a tubular neighborhood $\nu(L)$ of L such that

1. *The n-dimensional submanifold L is a Lagrangian torus, and $\nu(L)$ is diffeomorphic to $T^n \times D^n$.*
2. *There exists a coordinate $\{\phi, S\}$ on $\nu(L)$ such that*
   (a) *The symplectic form can be written as*
   $$\omega = \sum dS_i \wedge d\phi_i$$
   (b) *The coordinates S_i only depend on F_i.*
   (c) *Each Hamiltonian flow of F_i is linear in the coordinate.*

We say the Hamiltonian system (M, ω, H) is an **integrable system** if there exist $F_2, \cdots, F_n$ which satisfy the assumptions of Theorem 2.3.7. The coordinates introduced in the theorem are called the **action-angle coordinate**. The integrable system is deeply understood and studied system due to its simplicity.

Let (W, ω) be a symplectic manifold, $\tilde{H}$ be a Hamiltonian and $V \subset W$ be a symplectic submanifold. Then $H = \tilde{H}|_V$ also defines a Hamiltonian vector field field X_H on V, but it is not the same as the Hamiltonian vector field $X_{\tilde{H}}$ restricted to V. In particular, $X_{\tilde{H}}|_V$ is not tangent to V in general. Thus we need a projection formula for explicit computation.

Proposition 2.3.8. *Let W be a symplectic manifold, and $\tilde{H}$ be a Hamiltonian on W. Let $f, g : W \to \mathbb{R}$ be smooth functions and let c_1, c_2 be the regular values such that $V = f^{-1}(c_1) \cap g^{-1}(c_2) \subset W$ is a symplectic submanifold of W of codimension 2. Furthermore, assume that $\{f, g\}$, $\{f, \tilde{H}\}$, and $\{g, \tilde{H}\}$ are nonzero. Let $H = \tilde{H}|_V$. The Hamiltonian vector field X_H on V is given by*

$$X_H = X_{\tilde{H}} - \frac{\{g, \tilde{H}\}}{\{g, f\}} X_f - \frac{\{f, \tilde{H}\}}{\{f, g\}} X_g.$$

Proof. The submanifold V is contained in a level set of Hamiltonians f and g. Thus df and dg vanish on V, so do X_f and X_g. It follows that we can use X_f and X_g as normal directions of X_H.

Now we write X_H as $X_H = X_{\tilde{H}} + aX_f + bX_g$ for some functions a, b. Since X_H is defined on the level set of f and g, we must have $X_H(f) = 0 = X_H(g)$. We also have $X_f(f) = 0 = X_g(g)$ from the definition. It follows that

$$\begin{aligned} 0 &= X_{\tilde{H}}(f) + bX_g(f) = \{f, \tilde{H}\} + b\{f, g\}, \\ 0 &= X_{\tilde{H}}(g) + aX_f(g) = \{g, \tilde{H}\} + a\{g, f\}. \end{aligned}$$

Substituting a, b into the first equation yields the result. □

Remark 2.3.9. The formula of Proposition 2.3.8 can be generalized to the case of any $2k$-functions, say $V = f_1^{-1}(c_1) \cap \cdots \cap f_{2k}^{-1}(c_{2k})$. Following the same strategy, write $X_H = X_{\tilde{H}} + \sum a_i X_{f_i}$ and put f_j repeatedly, we get

$$0 = \{f_j, \tilde{H}\} + \sum_{i=1}^{2k} a_i \{f_j, f_i\}, \quad 1 \leq j \leq 2k.$$

The equation is linear with respect to a_i, and can be solved under some non-degeneracy assumptions.

2.3.3 Hamiltonian Actions

We refer to [Sep07] and [Hal15] for the basic notions of Lie groups and Lie algebras. Let (M, ω) be a symplectic manifold, G be a Lie group acting on M and $\mathfrak{g}$ be its Lie algebra. Then for any element $\xi \in \mathfrak{g}$, we get a path $\exp(t\xi)$ in G, which can be considered as a path of diffeomorphisms of M. We define the vector field X_ξ on M by

$$X_\xi(q) = \frac{d}{dt} \exp(t\xi) \cdot q.$$

If we require the action of G to be symplectic, which means that we have a group homomorphism

$$\Phi : G \to \mathrm{Symp}(M)$$

where $\mathrm{Symp}(M)$ is the group of symplectomorphisms of M, then the vector field X_ξ is a symplectic vector field. We further assume that each X_ξ is a Hamiltonian vector field, so there exists H_ξ such that $i_{X_\xi}\omega = dH_\xi$ for each $\xi \in \mathfrak{g}$. In this case, we have a map

$$\phi : \mathfrak{g} \to C^\infty(M).$$

The action is called **Hamiltonian** if ϕ is a Lie algebra homomorphism, which means that $H_{\{\xi,\eta\}} = \{H_\xi, H_\eta\}$.

Example 2.3.10. Let $SO(3)$ act on $T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$ by

$$A \cdot (x, y) = (Ax, Ay).$$

Since $A^tA = \mathrm{Id}$, the action is symplectic. Consider the corresponding Lie algebra $\mathfrak{so}(3)$ consists of skew-symmetric matrices. For $\xi \in \mathfrak{so}(3)$, we define a Hamiltonian by

$$H_\xi(x, y) = y^t \xi x.$$

The Hamiltonian equation is as follows.

$$\begin{aligned}\dot{x} &= \frac{\partial}{\partial y} H_\xi = \xi x \\ \dot{y} &= -\frac{\partial}{\partial x} H_\xi = -(y^t\xi)^t = -\xi^t y = \xi y.\end{aligned}$$

Solving this equation by taking the exponential, we can recover the $SO(3)$-action which was described above. In short, the $SO(3)$-action is Hamiltonian.

An important consequence of this definition is a celebrated theorem of Noether, relating symmetry and invariance of a Hamiltonian system.

Theorem 2.3.11 (Noether)**.** *Let (M, ω) be a symplectic manifold and G act on M Hamiltonianly. Let $H : M \to \mathbb{R}$ be a Hamiltonian which is G-invariant, which means $H(x) = H(g \cdot x)$ for any $g \in G$. Then for any $\xi \in \mathfrak{g}$, $\{H, H_\xi\} = 0$.*

Proof. Since H is G-invariant, we have $\{H, H_\xi\} = X_{H_\xi}(H) = 0$. □

There is a notion called the *moment map* which generalizes the Hamiltonian action. We refer to [MS17] for a detailed discussion of this topic.

2.3.4 Geodesic Flow as a Hamiltonian Flow

Let (M, g) be a complete Riemannian manifold. We defined a geodesic vector field X and a geodesic flow φ of (M, g) in Section 2.1.1. The metric g on TM induces the dual metric g^* on T^*M by the natural pairing, and we can define a (co-)geodesic flow and a (co-)geodesic vector field on T^*M.

The following description of geodesic flow as a Hamiltonian flow is based on [FvK18] Theorem 2.3.1. Let $q = q(t)$ be a geodesic and denote $\dot{q} = v$. We introduce the dual coordinate on T^*M,

$$p_i = g_{ij} v^j, \ v^i = g^{ij} p_j.$$

Lemma 2.3.12. *The equation of geodesic q on T^*M is*

$$\dot{p}_i + \frac{1}{2} g^{jk}_{,i} p_j p_k = 0$$

where $\dot{q}^i = g^{ij} p_j$.

Proof. Using the formula for Christoffel symbol, the geodesic equation

$$\dot{v}^a + \Gamma^a_{bc} v^b v^c = 0$$

is transformed into the following.

$$\partial_t (g^{ai} p_i) + \frac{1}{2} g^{ai} (g_{ib,c} + g_{ci,b} - g_{bc,i}) g^{bj} p_j g^{ck} p_k = 0$$

Differentiating the identity $g^{ij} g_{jk} = \delta^i_k$, we have $g^{ij}_{,l} g_{jk} + g^{ij} g_{jk,l} = 0$ for any l. It follows that

$$\begin{aligned} g^{ai} g_{ib,c} g^{bj} g^{ck} p_j p_k &= -g^{ai}_{,c} g_{ib} g^{bj} g^{ck} p_j p_k = -g^{ai}_{,c} \delta^j_i g^{ck} p_j p_k = -g^{aj}_{,c} g^{ck} p_j p_k, \\ g^{ai} g_{ci,b} g^{bj} g^{ck} p_j p_k &= -g^{ai}_{,b} g_{ci} g^{ck} g^{bj} p_j p_k = -g^{ai}_{,b} \delta^k_i g^{bj} p_j p_k = -g^{ak}_{,b} g^{bj} p_j p_k \\ -g^{ai} g_{bc,i} g^{bj} g^{ck} p_j p_k &= g^{bj}_{,i} g_{bc} g^{ai} g^{ck} p_j p_k = g^{bj}_{,i} g^{ai} \delta^k_b p_j p_k = g^{jk}_{,i} g^{ai} p_j p_k \end{aligned}$$

We get the formula

$$\Gamma^a_{bc} v^b v^c = -g^{aj}_{,b} g^{bk} p_j p_k + \frac{1}{2} g^{ai} g^{jk}_{,i} p_j p_k.$$

Also, we have

$$\partial_t (g^{ai} p_i) = g^{ai} \dot{p}_i + g^{ai}_{,b} v^b p_i = g^{ai} \dot{p}_i + g^{aj}_{,b} g^{bk} p_j p_k$$

To sum up, the geodesic equation has the form

$$g^{ai} \dot{p}_i + \frac{1}{2} g^{ai} g^{jk}_{,i} p_j p_k = 0.$$

Cancelling g^{ai} gives us the desired result. □

Proposition 2.3.13. *The geodesic vector field on T^*M is a Hamiltonian* *vector field with Hamiltonian*

$$H(q,p) := \frac{1}{2}||p||^2_{g^*}.$$

where T^*M is equipped with a canonical symplectic form $\omega = \sum dp_i \wedge dq_i$

Proof. We can write the Hamiltonian given in the statement as

$$H(q,p) = \frac{1}{2}(g^{jk}p_j p_k - 1).$$

By direct computation we have

$$dH = \frac{1}{2}g^{jk}_{,i}p_j p_k dq^i + g^{ij}p_j dp_i.$$

The Hamiltonian vector field is

$$X_H = -\frac{1}{2}g^{jk}_{,i}p_j p_k \partial_{p_i} + g^{ij}p_j\partial_{q_i},$$

which is the geodesic vector field given in Lemma 2.3.12. □

Note 2.3.14. Notice that Proposition 2.3.13 can be regarded as a generalization of a fact that motion of free particle in the Euclidean space can be described by a Hamiltonian flow of kinetic energy, which was mentioned in the end of Example 2.3.1.

Corollary 2.3.15. *Let M be a Riemannian manifold. The geodesic vector field is Reeb vector field of unit cotangent bundle ST^*M with respect to the contact form pdq.*

Proof. Note that $X = p\partial_p$ is transversal to $T^*_{\leq 1}M = H^{-1}(1/2)$, and $i_X\omega = pdq$ is a Liouville form. Thus we can apply Proposition 2.3.4 to see that the Hamiltonian vector field X_H is parallel to the Reeb vector field on ST^*M. Moreover, X_H is exactly the Reeb vector field if and only if $i_{X_H}\lambda = |p| = 1$, which is the case of unit cotangent bundle. □

Example 2.3.16. We've found the formula for the metric on the sphere under the stereographic projection in Example 2.1.8. We can write out the

Hamiltonian which defines the geodesic flow on S^n,

$$H(q,p) = \frac{1}{2}g(p,p) = \frac{2}{(1+|q|^2)^2}|p|^2.$$

We also note the Hamiltonian for the geodesic flow on T^*S^n. The stereographic projection $S^n \to \mathbb{R}^n$ can be extended to the cotangent bundles by formula

$$\begin{aligned} \Phi : T^*S^n &\to T^*\mathbb{R}^n \\ (x,y) &\mapsto \left(\frac{\vec{x}}{1-x_0}, (1-x_0)\vec{y} + y_0\vec{x} \right). \end{aligned}$$

The formula for the cotangent coordinates can be derived by preservation of the canonical 1-form $ydx = pdq$. We require the product of the lengths of dual vectors to be 1, and it follows that the metric of cotangent bundle of S^n under the stereographic projection can be written as

$$g_{ij} = \frac{(1+|q|^2)^2}{4}\delta_{ij}.$$

It follows that the Hamiltonian of the geodesic flow is

$$H(q,p) = \frac{1}{2}\left(\frac{1}{2}(1+|q|^2)|p|\right)^2.$$

2.4 Global Hypersurfaces of Section

In this section, we introduce the notion of global hypersurfaces of section and open book decompositions, which will be a main topic of Chapter 3. We refer to [FvK18] and [MvK22b] for the general discussion.

2.4.1 Global Hypersurfaces of Section

Let Y be a closed manifold, and X be a non-vanishing vector field on Y. A **global hypersurface of section** of X, or of its flow Fl^X, is an embedded submanifold $P \subset Y$ of codimension 1 with (possibly empty) boundary $\partial P = B$ such that

1. the vector field X is transverse to the interior $\mathring{P}$ of P,

2. the boundary B is X-invariant. In other words, X is tangent to B.

3. for each point q in Y, there exist positive numbers t_+, t_- such that

$$Fl^X_{t_+}(q), Fl^X_{-t_-}(q) \in P.$$

If only the first two condition hold, we call P a **(local) hypersurface of section**. If P is a global hypersurface of section, we can define the **(first) return time** τ_p for each $p \in \mathring{P}$ by

$$\tau_p = \min\{t > 0 : Fl^X_t(p) \in P\}$$

and the **(first) return map** by $\Psi(p) = Fl^X_{\tau_p}(p)$. The return map is a diffeomorphism.

Let $\dim W = n + 1$. If there exists a global hypersurface of section of (W, X), we can understand many features of the dynamics of vector field X, or equivalently time-dependent diffeomorphism Fl^X_t of $(n+1)$-dimensional manifold M by understanding the dynamics of diffeomorphism Ψ of n-dimensional manifold P. For example, the periodic orbit of X corresponds to the fixed point or periodic point of Ψ.

Example 2.4.1 (Birkhoff annulus)**.** The Birkhoff annulus, defined by Birkhoff [Bir13], is one of the oldest examples of the global hypersurface of section. Let S be a 2-sphere with positive curvature. It's known that there exists at least one closed geodesic γ on S, which we call an *equator*. Let ν be a unit normal vector field along γ. We define

$$P = \{(q, p) \in T^*S^2 \,:\, q \in \gamma,\, g(\nu_q, p) \geq 0\} \simeq S^1 \times [0, \pi].$$

The codimension 1 submanifold P is called the Birkhoff annulus, and is known to be a global hypersurface of section. This fact leads to the proof of the Poincaré's last geometric theorem.

Example 2.4.2. Consider the unit cotangent bundle of round sphere STS^n as a subset of $T^*\mathbb{R}^{n+1}$. Let $X = X_g$ be the unit geodesic vector field. Using the coordinates $(x, y) = (x_0, \cdots, x_n, y_0, \cdots, y_n) \in T^*\mathbb{R}^n$, we can write

$$ST^*S^n = \{(x, y) : |x|^2 = 1,\ |y|^2 = 1,\ \langle x, y\rangle = 0\}$$

Consider the submanifold

$$P = \{(x, y) \in ST^*S^n : x_0 = 0, y_0 \geq 0\},$$

which is the set of upward directions on the equator. We claim that P is a global hypersurface of section of X.

1. The geodesic with initial condition $x_0 = 0$ and $y_0 > 0$ leaves P, which means that X is transverse to P.

2. The boundary is the unit cotangent bundle of the equator S^{n-1}, which corresponds to $x_0 = 0$. This is totally geodesic, as we've seen in Example 2.1.12, which means that the boundary is X-invariant.

3. A geodesic on S^n is a great circle, as we've seen in Example 2.1.8, which means every geodesic eventually touches P.

Therefore P is a global hypersurface of section of X. We can easily see that the return map is the identity.

Example 2.4.3. Let M be an n-torus, defined by

$$M = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_j| = 1\}$$

and equipped with the standard metric. Let $N = \{z_n = 1\}$, then N is a totally geodesic submanifold. Let's identify ST^*M with $M \times S^{n-1}$ and consider a codimension 1 submanifold

$$P = \{(z, w) \in ST^*M\ :\ z \in N,\ w_n \geq 0\}.$$

If $(z, w) \in \mathring{P}$, we have $w_n > 0$, so the geodesic with the initial condition (z, w) escapes from $\mathring{P}$. Also, since $\partial P = ST^*N$ and N is totally geodesic, the geodesic vector field is tangent to ∂P. It follows that P is a local hypersurface of section of X.

However, P is not a global hypersurface of section. To see this, consider the geodesic with initial condition $\gamma(0) = (1, \cdots, 1, -1)$, $\dot{\gamma}(0) = (1, 0, \cdots, 0)$. Then we have

$$\gamma(t) = (e^{it}, 1, \cdots, 1, -1)$$

so γ never touches P, which means that the last condition is not satisfied.

2.4.2 Open Book Decompositions

An **open book decomposition** on a closed manifold Y is a pair (B, π) of a codimension 2 closed submanifold B and a map π from $Y \setminus B$ to $S^1 \subset \mathbb{C}$ which satisfies the following.

1. The normal bundle of B is trivial. We call B the **binding**. We fix the trivialization of tubular neighborhood $\xi : B \times D^2 \to \nu(B)$.

2. The map π is a fiber bundle such that $(\pi \circ \xi)(b; r, \theta) = e^{i\theta}$ on $\nu(B) \setminus B$, where (r, θ) is a polar coordinate on D^2. We call the closure of each fiber $\overline{\pi^{-1}(e^{i\theta})} = P_\theta$ the **page**. Note that $\partial P_\theta = B$ for any θ.

If X is transverse to each page P_θ and tangent to B, then we say X **is adapted to** (B, π).

Lemma 2.4.4. *Let Y be a closed manifold, $B \subset Y$ be a codimension 2 closed submanifold, X be a vector field on Y, and π be a map from $Y \setminus B$ to S^1. Assume that*

1. *For any $p \in Y$, $d_p\pi(X) > 0$,*

2. *There exists a trivial neighborhood $\nu(B) \simeq B \times D^2$ of B and with respect to the polar coordinate of D^2, $\pi(b; r, \theta) = e^{i\theta}$,*

3. *X is tangent to B.*

Then (B, π) is an open book decomposition of Y, which X is adapted to.

Proof. By the first condition, π is a submersion. We can remove an open tubular neighborhood $\nu(B)$ of B on which the second condition holds, then $\pi|_{Y \setminus \nu(B)}$ is still a submersion. In particular, the second condition guarantees that $\pi|_{\partial\nu(B)}$ is a submersion. Since $\pi|_{Y\setminus\nu(B)}$ is proper, we can apply the Ehresmann fibration theorem and conclude that π defines a fiber bundle. With (2), we can see that (B, π) is an open book decomposition on Y.

Since $d\pi(X) \neq 0$, X cannot be tangent to the level sets of π. This means that X is transverse to each page. With (3), we can see that X is adapted to (B, π). □

If a vector field X is adapted to (B, π), then each page $P_\theta = \pi^{-1}(e^{i\theta})$ can be regarded as a candidate for the global hypersurface of section. There might exist an orbit of X which does not return to the page in a finite time. Such an orbit should be asymptotic to the boundary as t becomes large. A discussion about a case of dimension 3 can be found in [HWZ98].

2.4.3 Global Hypersection of Sections of Reeb Flow

The system of our interest to find a global hypersurface of section and open book decomposition is a case of Hamiltonian flow and the Reeb flow. The relation of those two systems were revealed in Proposition 2.3.4. If we say P is a global hypersurface of section or (B, π) is an open book decomposition of a contact manifold $(Y, \ker\alpha)$, we assume P is a global hypersurface of section for the Reeb vector field R and (B, π) is adapted to the R unless otherwise mentioned.

Lemma 2.4.5. *Let $(Y, \xi = \ker\alpha)$ be a contact manifold. Assume that there exists a global hypersurface of section P on Y. Then,*

1. *The interior of P is a symplectic manifold with a symplectic form $d\alpha$.*
2. *The binding $(B, \xi_B = \xi|_{TB})$ is a contact submanifold of (Y, ξ).*
3. *The return map Ψ satisfies $\Psi^*\alpha - \alpha = d\tau$ where τ is the return time. In particular, Ψ is an exact symplectomorphism.*

Proof. For the first statement, let $\dim Y = 2n + 1$. Since Y is contact, $\alpha \wedge d\alpha^n$ never vanishes. Since $\mathring{P}$ is transverse to R, we can take a local frame $(X_1, \cdots, X_{2n})$ of $\mathring{P}$ such that $(R, X_1, \cdots, X_{2n})$ is a local frame of Y, and

$$\alpha \wedge d\alpha^n(R, X_1, \cdots, X_{2n}) = \alpha(R)d\alpha^n(X_1, \cdots, X_{2n}) \neq 0.$$

It means that $d\alpha$ is a non-degenerate closed 2-form on $\mathring{P}$, so it's a symplectic form. For the second statement, we can assume that (X_{2n-1}, X_{2n}) is the normal frame of B with respect to Y, since R is tangent to B. Then $(X_1, \cdots, X_{2n-2})$ is the basis of $\xi|_{TB}$ and we have

$$\alpha \wedge d\alpha^{n-1}(R, X_1, \cdots, X_{2n-2}) = \alpha(R) d\alpha^n(X_1, \cdots, X_{2n})/d\alpha(X_{2n-1}, X_{2n}) \neq 0.$$

Thus (B, ξ_B) is a contact manifold.

The third statement is a generalization of the well-known fact in dimension 3, which can be found, for example, in [ABHSa17], [FvK18] or [MvK22b] Lemma 5.4. We have $Fl^R_{\tau_p}(p) = \Psi(p)$. Differentiating both sides and plugging in a vector field X, we have

$$dFl^R_{\tau_p}(p)X + (d\tau_p(X))R = d_p\Psi(X).$$

Since Fl^R preserves α, we have that

$$\Psi^*\alpha(X) = \alpha(d\Psi(X)) = (Fl^R)^*\alpha(X) + \alpha(R)d\tau(X) = \alpha(X) + d\tau(X).$$

By differentiating both sides, we get $\Psi^* d\alpha = d\alpha$, which means that Ψ is a symplectomorphism. □

We also note some boundary behaviors. Let (B, π) be an open book decomposition of $(Y, \ker\alpha)$ and P be a page. Since $(\mathring{P}, d\alpha)$ is a symplectic manifold, it's natural to expect that $(P, d\alpha)$ is a Liouville domain. However, the following proposition says this never happens.

Proposition 2.4.6. *Let $(Y, \ker\alpha)$ be a contact manifold and (B, π) be its open book decomposition. Then $d\alpha$ degenerates on the boundary of a page P.*

Proof. The Reeb vector field R is tangent to ∂P, which means in particular $R_b \in T_bP$ for $b \in \partial P$. However, $i_R d\alpha = 0$ which means that $d\alpha$ degenerates on the boundary. □

Assume that the contact manifold $(Y, \ker\alpha)$ is given by a regular level set of a Hamiltonian H defined on a symplectic manifold (M, ω), and (B, π)

be an open book decomposition of $(Y, \ker\alpha)$. Let γ be a contractible Reeb orbit contained in B, and fix a symplectic normalization of $\xi = \ker\alpha$ along γ. We also take a symplectic normal frame (N_1, N_2) of normal bundle ν_B of B in Y along γ. [MvK22b] Section 8 says that there exists a Riemannian metric which decomposes the Hessian of H into block diagonal matrices. Precisely, we can write

$$\mathrm{Hess}(H) = \begin{pmatrix} S_\xi & 0 \\ 0 & S_\nu \end{pmatrix}$$

where $S_\xi \in \xi^* \otimes \xi^*$ and $S_\nu \in \nu_B^* \otimes \nu_B^*$. We call S_ν a **normal Hessian**. Practically, the normal Hessian can be computed by computing the linearized flow along the orbit contained in B.

Proposition 2.4.7 ([MvK22b])**.** *Under the above setting, we further assume that the return time of P is bounded. If S_ν is positive definite, then the return map Ψ extends to the boundary smoothly.*

Proof. See [MvK22b] Proposition 8.2. □

2.5 Floer Homology

In this section, we introduce various versions of Floer homology, including symplectic homology and S^1-equivariant symplectic homology. We also present the Morse-Bott spectral sequence, a useful tool for computing symplectic homology. For further details, we refer the reader to [AD14], [Abo15], [KvK16], and [Gut18].

2.5.1 Conley-Zehnder Index

Let Q be a quadratic form defined on the vector space V. In an appropriate basis, Q can be represented as a diagonal matrix with diagonal entries consisting only of 1, 0 and -1. Let n_+, n_0 and n_- denote the counts of 1 ,0 and -1 respectively. The **signature** of Q is defined as

$$\mathrm{Sign}(Q) = n_+ - n_-$$

Let $\Psi : [0,\tau] \to Sp(2n)$ be a path of symplectic matrices with $\Psi(0) = \mathrm{Id}$. A point $t \in [0,\tau]$ is called a **crossing** if $\det(\Psi(t) - \mathrm{Id}) = 0$. The **crossing form** is a quadratic form defined on the vector space $V_t = \ker(\Psi(t) - \mathrm{Id})$ as follows

$$Q_t(v,v) = \omega(v, \dot{\Psi}(t)v),$$

where ω is a symplectic form on V_t. Using a symplectic basis $\{v_1, w_1, \cdots, v_n, w_n\}$ as described in Example 2.2.6, the crossing form can be expressed as

$$Q_t = \Omega\dot{\psi}(t) = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\dot{\psi}(t)$$

Assuming the crossings are isolated, the Robbin-Salamon index (introduced in [RS93]) for Ψ is given by

$$\mu_{RS}(\psi) = \frac{1}{2}\mathrm{Sign}(Q_0) + \sum_t \mathrm{Sign}(Q_t) + \frac{1}{2}\mathrm{Sign}(Q_\tau)$$

where $\sum_t$ is the sum over all crossings.

The Robbin-Salamon index has the following key properties.

Theorem 2.5.1. *Let $\Psi_i : [0,\tau] \to Sp(2n)$ be paths. The Robbin-Salamon index μ_{RS} satisfies*

1. *(Homotopy Invariance) If Ψ_1 and Ψ_2 are homotopic,*

$$\mu_{RS}(\Psi_1) = \mu_{RS}(\Psi_2).$$

2. *(Additivity) Let $\Psi_3(t) = \Psi_2(t)\Psi_1(t)$ be the pointwise produce of Ψ_1 and a loop Ψ_2. Then,*

$$\mu_{RS}(\Psi_3) = \mu_{RS}(\Psi_2) + \mu_{RS}(\Psi_1).$$

Proof. For proofs and additional details, see [SZ92], [RS93], or [Sal99]. Note that the Robbin-Salamon index of a loop may differ from the Maslov index by a factor of two, depending on conventions. □

Let $(Y, \ker\alpha)$ be a contact manifold, R be the Reeb vector field and $\gamma : [0, \tau] \to Y$ be a periodic Reeb orbit. Assume that γ is contractible, so there exists a disk D_γ such that $\partial D_\gamma = \gamma$. Choosing a trivialization of the contact structure $A : \gamma^*\xi \to [0, \tau] \times \mathbb{R}^{2n}$, we obtain a path of symplectic matrices

$$\Psi(t) = A(t) dFl_t^R|_\xi A(0)^{-1} \in Sp(2n).$$

The **Conley-Zehnder index** of γ is defined as

$$\mu_{CZ}(\gamma) = \mu_{RS}(\Psi).$$

By the homotopy invariance property in Theorem 2.5.1, this definition does not depend on the choice of trivialization.

Note 2.5.2. In the case of geodesic flow, we can assign a Morse index to closed geodesics. From the perspective of Morse theory, closed geodesics are critical points of the energy functional on the loop space of the manifold. Details of this approach can be found in [Mil63].

The linearized geodesic flow coincides with the linearized Hamiltonian flow in this case and can be described using Jacobi fields. Consequently, the Morse index of a closed geodesic is equivalent to the Conley-Zehnder index of the same geodesic, viewed as a closed Reeb orbit.

Let $H : (W, \omega) \to \mathbb{R}$ be a Hamiltonian on a symplectic manifold and let $Y = H^{-1}(c)$ be a regular level set of contact type with Liouville vector field X. Let $\gamma : [0, \tau] \to Y$ be a Reeb orbit of $(Y, \ker i_X\omega)$. Since X_H is parallel to R, γ can be regarded as a Hamiltonian orbit with a different parametrization. Let $\tilde{\gamma}(s(t)) = \gamma(t)$, where $s(0) = 0$, $s(\tau) = \sigma$. Denote the initial and final points as $\gamma(0) = \tilde{g}(0) = q_0$ and $\gamma(\tau) = \tilde{\gamma}(\sigma) = q_1$. Linearized flows satisfy

$$dFl_\sigma^{X_H} : T_{q_0}W \to T_{q_1}W$$
$$dFl_\tau^R : T_{q_0}Y \to T_{q_1}Y.$$

Lemma 2.5.3. *Let X be a vector field on a manifold M, and $s : M \to \mathbb{R}$.*

Define $\psi(q) = Fl^X_{s(q)}(q)$. *Then,*

$$d\psi(q)\xi = dFl^X_{s(q)}(q)\xi + (ds(q)\xi)X.$$

Proof. See Chapter 9 of [FvK18]. □

Let N be a normal vector to Y in W. Then we have

$$T_qW = \langle N_q \rangle \oplus T_qY = \langle N_q \rangle \oplus \langle R_q \rangle \oplus \xi_q.$$

Consider the map given by quotient

$$dFl^{X_H}_\sigma : T_{q_0}Y \to T_{q_1}Y.$$

From Lemma 2.5.3, the difference between $dFl^{X_H}_\sigma$ and dFl^R_τ is parallel to R. This implies that, after quotienting by $\langle R \rangle$, the two maps

$$\begin{aligned} dFl^{X_H}_\sigma &: \xi_{q_0} \to \xi_{q_1} \\ dFl^R_\tau &: \xi_{q_0} \to \xi_{q_1} \end{aligned}$$

are identical. We can summarize this result as follows.

Proposition 2.5.4. *Let* $H : (M, \omega) \to \mathbb{R}$ *be a Hamiltonian, and let* $Y = H^{-1}(c)$ *be a regular level set of contact type. Let* γ *be a Reeb orbit, and let* $\tilde{\gamma}(s(t)) = \gamma(t)$ *be its corresponding Hamiltonian orbit. Then,*

$$dFl^R_t|_\xi = dFl^{X_H}_{s(t)}|_\xi$$

In particular, the Conley-Zehnder index can be computed using the linearized Hamiltonian flow.

2.5.2 Floer Homology

Floer homology, introduced by Floer [Flo89], is an important invariant of symplectic manifolds. The chain complex is generated by periodic Hamiltonian orbits, while the differential counts Floer cylinders. This can be regarded as a Morse homology defined on the loop space. For aspherical closed symplectic manifolds, Floer homology is known to be isomorphic to singular homology, a result that provides a proof of Arnold's conjecture. In this subsection, we briefly introduce Floer homology and symplectic homology without delving into detailed constructions. For further details, see [AD14].

et (W, ω) be a closed symplectic manifold such that $\pi_2(W) = 0$, and let $H : W \times S^1 \to \mathbb{R}$ be a time-dependent Hamiltonian. Let J be an almost complex structure on W, compatible with ω. For a contractible loop γ in W, let D be a capping disk of γ, meaning $D : D^2 \to W$ with $D|_{\partial D^2} = \gamma$. We define the **action functional** on the loop space ΛW of W by

$$\mathcal{A}(\gamma) = -\int_D u^*\omega + \int_0^1 H_t(\gamma(t))dt.$$

The critical points of $\mathcal{A}$ are 1-periodic Hamiltonian orbits of H. We call an orbit γ is **nondegenerate** if

$$\det(\mathrm{Id} - d_{\gamma(0)} Fl_1^{X_H}) \neq 0.$$

Note that γ is nondegenerate as a Hamiltonian orbit if and only if it is nondegenerate as a critical point of $\mathcal{A}$ in the Morse-theoretic sense. We further assume that every Hamiltonian orbit of H is nondegenerate. This condition can be generically achieved by perturbing the Hamiltonian.

We define the **Floer chain group** as a graded $\mathbb{Z}_2$-vector space

$$CF_*(W, H, J) = \bigoplus_{\gamma \in \mathrm{Crit}(\mathcal{A})} \mathbb{Z}_2 \cdot \gamma$$

where the degree of γ is given by its Conley-Zehnder index. Let γ_+, γ_- be 1-periodic Hamiltonian orbits. Define the moduli space of **Floer cylinders**,

$$\mathcal{M}(\gamma_-, \gamma_+, H, J) = \left\{ u : S^1 \times \mathbb{R} \to W \ : \ \begin{array}{c} u_s + J(u_t - X_{H_t}) = 0, \\ \lim_{s \to \pm\infty} u(s, -) = \gamma_\pm(-) \end{array} \right\}.$$

For generic J, it is well-known that $\mathcal{M}(\gamma_-, \gamma_+, H, J)$ is smooth manifold of dimension

$$\dim \mathcal{M}(\gamma_-, \gamma_+, H, J) = \mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) + 1.$$

The differential on $CF_*(W, H, J)$ is defined by

$$\partial_*^F(\gamma_+) = \sum_{\mu_{CZ}(\gamma_-)=*-1} \#\mathcal{M}(\gamma_-, \gamma_+, H, J) \cdot \gamma_-$$

Using a gluing argument, one can show that $(\partial^F)^2 = 0$. Hence, (CF^*, ∂^F) forms a chain complex, which defines the **Floer homology** $HF_*(W, H, J)$. It is well-known that $HF_*(W, H, J)$ is independent of the choice of H and J.

The invariance of Floer homology is shown using a *continuation map*, which is a map from $HF_*(W, H_1, J_1)$ to $HF_*(W, H_2, J_2)$. In particular, we have the following result.

Theorem 2.5.5. $HF_*(W) \simeq H_*^{\mathrm{Morse}}(W, \mathbb{Z}_2)$.

Idea of the Proof. Take H small enough so that every Hamiltonian 1-orbit is a critical point of H. In particular, we can take H can be chosen as a Morse function. Then, there is a correspondence between generators, and one can show a correspondence between differentials. □

2.5.3 Symplectic Homology

For open symplectic manifolds, one can use symplectic homology, as defined in [FH94], [CFH95]. We consider Liouville domains with Hamiltonians that behave well on the boundary, so periodic Reeb orbits on the boundary of the manifold also serve as generators. A celebrated result, Viterbo's theorem [AS06], [Abo15], states that in the case of a cotangent bundle T^*M, the symplectic homology is isomorphic to the homology of the free loop space of M. See [Abo15] for details.

Consider a Liouville domain $(W, d\lambda)$ with a Liouville vector field X. Let $\alpha = \lambda|_{\partial W}$ be the contact form on ∂W. The flow of X provides a trivialization of the collar neighborhood of ∂W given by

$$\begin{aligned}\Phi_X : (0,1] \times \partial W &\to W \\ (r,q) &\mapsto Fl^X_{\log r}(q).\end{aligned}$$

Note that Φ_X identifies $r\alpha$ with λ. We can complete $(W, d\lambda)$ by attaching a *symplectization* of ∂W, defined as

$$\hat{W} = W \sqcup_{\Phi_X} (0,\infty) \times \partial W$$

equipped with the Liouville form $\hat{\lambda}$, where $\hat{\lambda}|_W = \lambda$ and $\hat{\lambda}|_{(0,\infty)\times\partial W} = r\alpha$. We impose the following conditions on the Hamiltonian H and the almost complex structure J on $S^1 \times \hat{W}$:

1. The Hamiltonian H is **linear at infinity**. Specifically, there exists a number $\tau > 0$, distinct from the period of any closed Reeb orbit of $(\partial W, \ker \alpha)$, such that outside a compact set,

$$H(t,x,r) = \tau r - C$$

   for some constant C. Additionally, we assume $H < 0$ inside W.

2. J is **cylindrical at infinity**. This means that J preserves the contact form on each $\partial W \times \{r\}$, satisfies $JX = R$ where R is the Reeb vector field, and J is invariant under translation in the $\mathbb{R}$-component.

With these conditions, we can define Floer homology $HF_*(H,J)$. It is well-known that this group is independent of the choice of a cylindrical almost complex structure, so we write it as $HF_*(H)$. If the slope of H is τ, the chain complex $CF_*(H)$ is generated by the interior Hamiltonian orbits and closed Reeb orbits on ∂W with periods less than τ.

We define a partial ordering on Hamiltonians that are linear at infinity by

$$H_1 \prec H_2 \text{ if and only if } H_1(t,x) \le H_2(t,x) \text{ for any } (t,x) \in S^1 \times \hat{W}.$$

If $H_1 \prec H_2$, we can define a continuation map. With this, we define **sym-**

plectic homology by

$$SH_*(W,\lambda) = \varinjlim_{(\mathcal{H},\prec)} HF_*(H)$$

We also define the filtered version of symplectic homology $SH_*^a(W,\lambda)$. The filtration is given by the action of the orbit, which equals the period in the case of a Reeb orbit. The homology $SH_*^a(W,\lambda)$ consists of Reeb orbits with periods less than a, and is isomorphic to the Floer homology $HF_*(H)$, where H is a Hamiltonian with slope a. In particular, take $\varepsilon > 0$ small enough such that ε is less than the shortest period of Reeb orbits on ∂W. Then $SH_*^\varepsilon(W,\lambda)$ consists only of interior Hamiltonian orbits as generators, giving

$$SH_*^\varepsilon(W,\lambda) \simeq H_*(W,\partial W).$$

We define the **+-part of symplectic homology** by

$$SH_*^+(W,\lambda) = SH_*(W,\lambda)/SH_*^\varepsilon(W,\lambda).$$

This homology group consists only of Reeb orbits as generators.

Theorem 2.5.6 (Viterbo's Theorem)**.** *The symplectic homology of the cotangent bundle of a manifold M is isomorphic to the singular homology of the free loop space ΛM.*

Proof. See [AS06] or [Abo15]. □

2.5.4 S^1-equivariant Symplectic Homology

The S^1-equivariant version of symplectic homology was introduced in [Vit99], [Sei08], and further developed in [BO17]. In contrast, S^1-equivariant symplectic homology, these are identified as a single generator. In this sense, S^1-equivariant symplectic homology simplifies the chain complex and provides more direct intuition. For details, refer to the lecture notes of Gutt [Gut18].

Let (W,ω) be a Liouville domain, and let $\hat{W}$ be its completion. Define the Morse function $f_N : \mathbb{CP}^N \to \mathbb{R}$ by

$$f_N([w_0 : \cdots : w_n]) = \frac{\sum_j j|w_j|^2}{\sum |w_j|^2}.$$

This function has $(N+1)$-critical points with degree $0, 2, \cdots, 2N$. Let $\tilde{f}_N : S^{2N+1} \to \mathbb{R}$ be the lift of f_N via S^1-fibration $S^{2N+1} \to \mathbb{CP}^N$. For each critical value of $\tilde{f}_N$, there exists an S^1-family of critical points of $\tilde{f}_N$.

Now consider S^1-equivariant parametrized Hamiltonian

$$H : S^1 \times \hat{W} \times S^{2N+1} \to \mathbb{R}$$

such that $H(\theta, x, z) = H(\theta + \varphi, x, \varphi \cdot z)$. Assume the following:

1. The Hamiltonian H is linear at infinity.
2. If $z \in \mathrm{Crit}(\tilde{f}_N)$, then the 1-periodic orbits of X_{H_z} are nondegenerate.
3. Along the negative gradient flow of $\tilde{f}_N$, H is nondecreasing.

Let $\mathcal{P}(H, N)$ denote the set of pairs (γ, z), where $z \in \mathrm{Crit}(\tilde{f}_N)$ and γ is a 1-periodic orbit of H_z. There is a free S^1-action on $\mathcal{P}(H, N)$ defined by

$$\varphi \cdot (\gamma, z) = (\gamma(\cdot - \varphi), \varphi z).$$

We denote the S^1-orbit of $p = (\gamma, z)$ in $\mathcal{P}(H, N)$ by S_p. Let $J = J_z^\theta$ be an S^1-equivariant generic almost complex structure parametrized by $S^1 \times S^{2N+1}$.

Let $p_\pm$ be elements of $\mathcal{P}(H, N)$. The moduli space of Floer trajectories is defined as

$$\mathcal{M}(S_{p_+}, S_{p_-}) = \left\{ \begin{array}{c} \eta : \mathbb{R} \to S^{2N+1} \\ u : \mathbb{R} \times S^1 \to \hat{W} \end{array} : \begin{array}{c} \dot{\eta} + \nabla \tilde{f}_N(\eta) = 0 \\ u_s + J^\theta_{\eta(s)}(u_\theta - X_{H^\theta_{\eta(s)}}) = 0 \\ \lim_{s \to \pm\infty}(\eta(s), u(s, -)) \in S_{\gamma_\pm, z_\pm} \end{array} \right\}$$

There exists a natural $\mathbb{R}$- and S^1-action defined on $\mathcal{M}$, and we denote the quotient space by $\mathcal{M}^{S^1}$. For $p = (\gamma, z)$, let $|S_p| = \mathrm{Ind}(z) + \mu_{CZ}(\gamma)$. For a generic J, $\mathcal{M}^{S^1}$ is a smooth manifold with dimension

$$\dim \mathcal{M}^{S^1}(S_{p_+}, S_{p_-}) = |S_{p_+}| - |S_{p_-}| - 1.$$

where $\mathrm{Ind}(z)$ is the Morse index z as a critical point of f_N. We define the chain complex $CF_*^{S^1,N}(H,J)$ as the $\mathbb{Z}_2$-vector space generated by S_p's, with differential ∂^{S^1} given by

$$\partial^{S^1}(S_{p_+}) = \sum_{|S_{p_-}|=|S_{p_+}|+1} \#\mathcal{M}^{S^1}(S_{p_+}, S_{p_-})S_{p_-}.$$

It is known that $(\partial^{S^1})^2 = 0$, allowing us to define $HF_*^{S^1,N}$.

There exists a natural inclusion of $S^{2N+1} \subset \mathbb{C}^{N+1}$ into $S^{2N+3} \subset \mathbb{C}^{N+2}$ for any N, given by $z \mapsto (z, 0)$. If H is defined on S^{2N+3}, we can pull-back H to S^{2N+1}. This allows the definition of a partial ordering on the set of pairs (H, N) of an appropriate Hamiltonian H and a natural number N, as in the symplectic homology. We define **S^1-equivariant symplectic homology** as

$$SH_*^{S^1}(W,\lambda) = \varinjlim_{(H,N)} HF_*^{S^1,N}(H).$$

2.5.5 Morse-Bott Spectral Sequence

One way to compute the symplectic homology of spaces with sufficient symmetry is by using the Morse-Bott spectral sequence. If periodic orbits are degenerate and form a submanifold, we can use local Floer homology to construct a spectral sequence. For details, see [KvK16].

Let $(W, d\lambda)$ be a Liouville domain and H be an autonomous Hamiltonian defined on W. Consider the critical manifold of 1-periodic orbits of X_H,

$$C = \{x \in W \,:\, Fl_1^{X_H}(x) = x\}.$$

We assume that C is compact manifold without boundary. Let Σ be a connected component of C. We say Σ is of **Morse-Bott type** if the linearized return map restricted to the normal bundle of Σ is nondegenerate, i.e.,

$$\det\left(d_x Fl_1^{X_H}|_{\nu(\Sigma)} - \mathrm{Id}|_{\nu(\Sigma)}\right) \neq 0.$$

Assume H is Morse-Bott, meaning every component of C is of Morse-Bott type. By perturbing H near Σ using a Morse function on Σ, we can define **local Floer homology** $HF_*^{loc}(\Sigma)$ of Σ. Under certain assumptions outlined in Proposition 8.4 of [KvK16], which are satisfied in our case, there is an isomorphism

$$\begin{aligned} HF^{\text{loc}}_{*+sh(\Sigma)}(\Sigma, H, J) &\simeq H_*^{\text{Morse}}(\Sigma, \mathbb{Z}_2), \\ sh(\Sigma) &= \mu_{RS}(\Sigma) - \frac{1}{2}\dim \Sigma/S^1 \end{aligned}$$

where $sh(\Sigma)$ is the shift term. Using this, we can define the **Morse-Bott spectral sequence**, where the filtration is given by the action functional.

Theorem 2.5.7 ([KvK16])**.** *Under appropriate assumptions,*

1. *There exists a spectral sequence converging to $SH(W)$, with the E^1-page given by*

$$E^1_{pq}(SH) = \begin{cases} \bigoplus_{\Sigma \in C} H_{p+q-sh(\Sigma)}(\Sigma, \mathbb{Z}_2) & p > 0 \\ H_{q+n}(W, \partial W, \mathbb{Z}_2) & p = 0 \\ 0 & p < 0 \end{cases}$$

2. *There exists a spectral sequence converging to $SH^+(W)$, with the E^1-page given by*

$$E^1_{pq}(SH) = \begin{cases} \bigoplus_{\Sigma \in C} H_{p+q-sh(\Sigma)}(\Sigma, \mathbb{Z}_2) & p > 0 \\ 0 & p \leq 0 \end{cases}$$

3. *There exists a spectral sequence converging to $SH^{S^1}(W)$, with the E^1-*

page given by

$$E^1_{pq}(SH) = \begin{cases} \bigoplus_{\Sigma \in C} H^{S^1}_{p+q-sh(\Sigma)}(\Sigma, \mathbb{Z}_2) & p > 0 \\ H^{S^1}_{q+n}(W, \partial W, \mathbb{Z}_2) & p = 0 \\ 0 & p < 0 \end{cases}$$

4. *There exists a spectral sequence converging to* $SH^{S^1,+}(W)$, *with the* E^1*-page given by*

$$E^1_{pq}(SH) = \begin{cases} \bigoplus_{\Sigma \in C} H^{S^1}_{p+q-sh(\Sigma)}(\Sigma, \mathbb{Z}_2) & p > 0 \\ 0 & p \leq 0 \end{cases}$$

Chapter 3

Existence of Global Hypersurfaces of Section

In this chapter, we prove the existence of global hypersurfaces of section for two cases. The first case is a certain type of Hamiltonian flow in $T^*\mathbb{R}^n$ and will be discussed in Section 3.1. This case includes the classical mechanical Hamiltonian with a convex potential, particularly the harmonic oscillator and the Hénon-Heiles system, as well as the contact ellipsoids.

The second case is the geodesic flow of a convex hypersurface contained in Euclidean space and will be discussed in Section 3.2. This part is based on joint work with Sunghae Cho [CL24]. This case includes the geodesic flow on a hypersurface of revolution, which includes a specific type of ellipsoid. In this case, we will compute the return map explicitly. Another example is the spatial Kepler problem which will be discussed in Chapter 4.

3.1 Mechanical Hamiltonian Flow

We state the main theorem of this section.

Theorem 3.1.1. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a symmetric mechanical Hamiltonian of a convex type, and c be a regular value. Then, there exists a global* *hypersurface of section*

$$P = \left\{ (q,p) \in H^{-1}(c) \, : \, q_1 = 0, \, p_1 \geq 0 \right\}$$

of the Hamiltonian flow X_H. Furthermore, the return map extends smoothly to the boundary.

The definition of a symmetric mechanical Hamiltonian of a convex type will be introduced in the following section.

3.1.1 Mechanical Hamiltonian of Convex Type

Let $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ be a Euclidean space equipped with the standard symplectic form $dp \wedge dq$. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian of the form

$$H(q,p) = W(p) + V(q).$$

Let c be a regular value of H. We first assume the following:

(A0) For any p such that $(q,p) \in H^{-1}(c)$, $p \cdot \nabla W > 0$.

This condition implies that $p \cdot \partial_p$ is transversal to $H^{-1}(c)$. Thus, it's a Liouville vector field and $H^{-1}(c)$ is a contact manifold with the contact form pdq. Instead, we might take the condition $q \cdot \nabla V > 0$. In this case, $q \cdot \partial_q$ is transversal to $H^{-1}(c)$ and the contact form is $-qdp$.

Here is the symmetry condition of H.

(A1) There exists a reflection R along a hyperplane containing the origin in $\mathbb{R}^n$, such that $V(q) = V(Rq)$ and $W(p) = W(Rp)$.

After an appropriate Euclidean transformation, we can assume that the reflection is given by

$$R(q_1, q_2, \cdots, q_n) = (-q_1, q_2, \cdots, q_n).$$

With this convention, we assume further:

(A2) $\partial^2_{q_1} V > 0$, $\partial^2_{p_1} W > 0$.

The condition (A2) can be replaced by the following two assumptions,

1. $q_1\partial_{q_1}V(q) > 0$ and $p_1\partial_{p_1}W(p) > 0$ for any $q_1 \neq 0$.
2. $\partial^2_{q_1}V(0, q_2, \cdots, q_n) > 0$ and $\partial^2_{p_1}W(0, q_2, \cdots, q_n) > 0$.

Indeed, from (A1), we have $\partial_{q_1}V(0, \vec{q}) = 0$ for any $\vec{q}$. From (A2), we deduce that $\partial_{q_1}V > 0$ if $q_1 > 0$, and $\partial_{q_1}V < 0$ if $q_1 < 0$. Thus, (A2) implies these two conditions, which are sufficient to prove the theorem.

For given H and c, **Hill's regions** are defined by

$$\mathcal{H}^q_c = \mathrm{pr}_1(H^{-1}(c)) \subset \mathbb{R}^n, \qquad \mathcal{H}^p_c = \mathrm{pr}_2(H^{-1}(c)) \subset \mathbb{R}^n,$$

where $\mathrm{pr}_1(q,p) = q$, $\mathrm{pr}_2(q,p) = p$. These regions represent the maximal set of possible values for q and p at a given energy level c. We further assume the following:

(A3) Hill's regions $\mathcal{H}^q_c$ and $\mathcal{H}^p_c$ are compact.

If (A0), (A1), (A2) and (A3) are satisfied, we call H a **symmetric mechanical Hamiltonian of convex type**.

3.1.2 Proof of the Existence

We denote $Y = H^{-1}(c)$ for a regular value c. We have

$$X_H = \nabla W \cdot \partial_q - \nabla V \cdot \partial_p$$

where $\nabla V = (\partial_{q_1}V, \cdots, \partial_{q_n}V)$ and $\nabla W = (\partial_{p_1}W, \cdots, \partial_{p_n}W)$. We define

$$B = \{(q,p) \in Y \ : \ q_1 = p_1 = 0\}.$$

Lemma 3.1.2. *The submanifold $B \subset Y$ has a trivial normal bundle.*

Proof. The global normal frame is given by $\partial_{p_1}, \partial_{q_1}$. □

Lemma 3.1.3. *Under assumption (A1), $B \subset Y$ is tangent to X_H.*

Proof. We have

$$X_H|_{TB} = \sum_{i=2}^{n} \frac{\partial W}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_{i=2}^{n} \frac{\partial V}{\partial q_i}\frac{\partial}{\partial p_i}$$

since (A1) implies that $\partial_{q_1} V(q) = 0$ and $\partial_{p_1} W(p) = 0$ on TB. □

We define a fibration map by

$$\begin{aligned} \pi : Y \setminus B &\to S^1 \subset \mathbb{C} \\ (q,p) &\mapsto \frac{q_1 + ip_1}{|q_1 + ip_1|}. \end{aligned}$$

The **angular form** is defined by

$$\Theta = i \cdot d \log \pi = \frac{p_1 dq_1 - q_1 dp_1}{q_1^2 + p_1^2} = \frac{\theta}{q_1^2 + p_1^2}.$$

Then we have

$$\theta(X_H) = p_1 \partial_{p_1} W + q_1 \partial_{q_1} V.$$

Lemma 3.1.4. *If H is a symmetric mechanical Hamiltonian of a convex type, there exists $\varepsilon > 0$ such that $\Theta(X_H) > \varepsilon$ for any $(q,p) \in Y \setminus B$.*

Proof. First, from (A2) and (A3), there exists $\eta_1 > 0$ such that $\partial_{q_1}^2 V(0, \vec{q}) > \eta_1$ for any $(0, \vec{q}) \in \mathrm{pr}_1(B)$. From (A1), we have $\partial_{q_1} V = 0$ along B. So the Taylor expansion of V with respect to q_1 along B is given by

$$\begin{aligned} V(q_1, \vec{q}) &= V(0, \vec{q}) + \partial_{q_1} V(0, \vec{q}) q_1 + \frac{1}{2}\partial_{q_1}^2 V(0, \vec{q}) q_1^2 + O(q_1^3) \\ &= V(0, \vec{q}) + \frac{1}{2}\partial_{q_1}^2 V(0, \vec{q}) q_1^2 + O(q_1^3), \\ \partial_{q_1} V(q_1, \vec{q}) &= \partial_{q_1}^2 V(0, \vec{q}) q_1 + O(q_1^2) \end{aligned}$$

where $\vec{q} = (q_2, \cdots, q_n)$. Take $\delta > 0$ such that if $|q_1| < \delta$, then

$$\left| \frac{\partial_{q_1} V(q_1, \vec{q})}{q_1} - \partial_{q_1}^2 V(0, \vec{q}) \right| < \frac{\eta_1}{2}.$$

Similarly, we can bound W, so it follows that if $|p_1|, |q_1| < \delta$,

$$\theta(X_H) = p_1^2 \frac{\partial_{p_1} W}{p_1} + q_1^2 \frac{\partial_{q_1} V}{q_1} > \frac{\eta_1}{2}(p_1^2 + q_1^2)$$

Now by (A2) and (A3), there exists $\eta_2 > 0$ such that if $|q_1| > \delta$,

$$\frac{\partial_{q_1} V}{q_1} = \frac{1}{q_1^2} \cdot q_1 \partial_{q_1} V > \eta_2.$$

Again, we take a similar bound for W, and it follows that if $|p_1|, |q_1| > \delta$,

$$\theta(X_H) = p_1 \partial_{p_1} W + q_1 \partial_{q_1} V > \eta_2(p_1^2 + q_1^2).$$

Take $\varepsilon = \min(\eta_1/2, \eta_2)$, and the result follows. □

Theorem 3.1.5. *$\pi : Y \setminus B \to S^1$ defines an open book decomposition of Y, to which X_H is adapted.*

Proof. The three assumptions of Lemma 2.4.4 were proved in Lemma 3.1.4, Lemma 3.1.2 and Lemma 3.1.3, so we can apply the lemma. □

Theorem 3.1.6. *Each page of the open book decomposition (B, π) given in Theorem 3.1.5 is a global hypersurface of section.*

Proof. We only need to demonstrate that the return time is bounded. It suffices to show that there exists $t \in \mathbb{R}_{>0}$ such that $\pi(Fl_t^{X_H}(q,p)) = \pi(q,p)$. From Lemma 3.1.4, we observe the existence of $\varepsilon > 0$ such that $\Theta(X_H) > \varepsilon$. Thus, we have

$$\int_0^{2\pi/\varepsilon} i \cdot d \log \pi(X_H) > \int_0^{2\pi/\varepsilon} \varepsilon dt > 2\pi.$$

By the intermediate value theorem, we can conclude that there exists a positive number $\tau < 2\pi/\varepsilon$ such that $\int_0^\tau i \cdot d \log \pi(X_H) = 2\pi$, which means that $\pi(Fl_\tau^{X_H}(q,p)) = \pi(q,p)$. It means that τ is a bounded positive finite return time for (q,p). The boundedness of the negative return time can be demonstrated similarly. □

3.1.3 Extension of Return Map

To complete the proof of Theorem 3.1.1, we consider the normal Hessian. By differentiating the Hamiltonian equation, the linearized flow is given by

$$L = \begin{pmatrix} 0 & -\mathrm{Hess}(V) \\ \mathrm{Hess}(W) & 0 \end{pmatrix}$$

with respect to the frame (∂_p, ∂_q). Under the normal frame $(\partial_{p_1}, \partial_{q_1})$, we have

$$L = \begin{pmatrix} 0 & -\partial_{q_1}^2 V \\ \partial_{p_1}^2 W & 0 \end{pmatrix}$$

The normal Hessian is obtained as $-JL$, where J is the almost complex structure defined on the symplectic normal bundle. Thus,

$$S_\nu = \begin{pmatrix} \partial_{q_1}^2 V & 0 \\ 0 & \partial_{p_1}^2 W \end{pmatrix}.$$

By (A2), S_ν is positive definite, so we have the result.

Theorem 3.1.7. *The return map of the global hypersurface of section given in Theorem 3.1.6 extends smoothly to the boundary.*

Proof. This follows from Proposition 2.4.7. □

3.1.4 Examples

Example 3.1.8. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be given by

$$H(q,p) = \frac{1}{2}|p|^2 + \frac{1}{2}k|q|^2.$$

for $k > 0$. This is the Hamiltonian of the **harmonic oscillator**. We can directly see that H is a mechanical Hamiltonian of a convex type, and apply Theorem 3.1.1. Actually, $V(q) = \frac{1}{2}k|q|^2$ is symmetric with respect to every reflection along a hyperplane contains the origin. That is, for any nonzero vector ν and $c > 0$,

$$P_\nu = \left\{(q,p) \in H^{-1}(c) \,:\, \nu \cdot q = 0,\, \nu \cdot p \geq 0\right\}$$

is a global hypersurface of section on $H^{-1}(c)$.

If we take $\nu = (1, 0, \cdots, 0)$ and identify $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ by $(q,p) \mapsto q + ip$, we can see that $H^{-1}(c)$ is diffeomorphic to S^{2n-1}, P is given by $\mathrm{Im}(z_1) \geq 0$, and the binding $B = \partial P$ is the equator given by $z_1 = 0$.

Example 3.1.9. Consider the Hamiltonian

$$H(q,p) = \sum a_i p_i^2 + \sum b_i q_i^2$$

for $a_i, b_i > 0$. The level set is an ellipsoid. For any i and $c > 0$, we have the global hypersurface of sections P_i, Q_i for $i = 1, \cdots, n$ given by

$$P_i = \left\{(q,p) \in H^{-1}(c) \,:\, q_i = 0,\, p_i \geq 0\right\}$$
$$Q_i = \left\{(q,p) \in H^{-1}(c) \,:\, q_i \geq 0,\, p_i = 0\right\}$$

Example 3.1.10. The **Hénon–Heiles system** is defined by the Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 + V(q) = \frac{1}{2}|p|^2 + \frac{1}{2}|q|^2 + (q_1^2 + q_2^2)q_3 - \frac{q_3^3}{3}.$$

This system was introduced by Hénon and Heiles [HH64] and describes the galactic dynamics. It's known that this system is not integrable, and is chaotic. See [For91], [FLP$^+$98a] and [FLP$^+$98b] for details.

The level contour of the potential V is illustrated in Figure 3.1[1], and we can see that the Hill's region is compact for energy level $c < 1/6$. In particular, $q_3 > -1/2$ if $c < 1/6$. We can see that $V(q_1, q_2, q_3) = V(-q_1, q_2, q_3)$ and $\partial_{q_1}^2 V = 1 + 2q_3 > 0$, so we can apply Theorem 3.1.1. The global hypersurface of section is given by the planar problem,

$$P = \left\{(q,p) \in H^{-1}(c) \,:\, q_1 = 0,\, p_1 \geq 0\right\}.$$

[1] https://commons.wikimedia.org/wiki/File:Henon_heiles_potential.svg

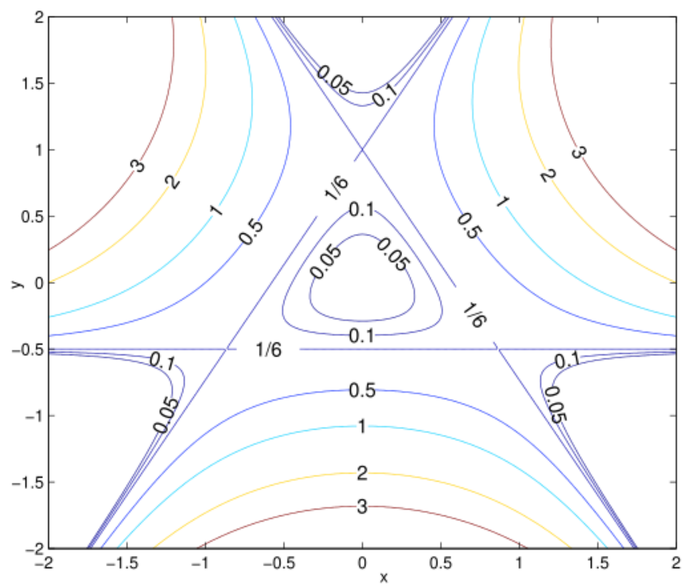

Figure 3.1: [Deb12] The level contour of the potential of the Hénon-Heiles system. The horizontal axis corresponds to $r = \sqrt{q_1^2 + q_2^2}$ and the vertical axis corresponds to q_3.

3.2 Geodesic Flow on Convex Hypersurfaces

We first state the main result of this section.

Theorem 3.2.1 ([CL24], Theorem 1.1.). *Let $M \subset \mathbb{R}^{n+1}$ be a locally symmetric convex hypersurface with fixed locus N. Then the geodesic flow on ST^*M admits a global hypersurface of section*

$$P = \{(x, y) \in ST^*M : x \in N, \langle y, \nu_x \rangle \geq 0\},$$

where ν is a normal vector field of N with respect to M. Moreover, the return map extends smoothly to the boundary of P.

A locally symmetric convex hypersurface will be defined in the following section.

3.2.1 Geodesic Flows on Hypersurfaces

For simplicity, we denote $x = (x_0, \vec{x})$ for points in $\mathbb{R}^{n+1}$ and $y = (y_0, \vec{y})$ for vectors in $T_x\mathbb{R}^{n+1}$. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function with 0 as a regular value, and let $M = f^{-1}(0)$. We can embed T^*M into $T^*\mathbb{R}^{n+1}$ as

$$T^*M = \left\{(x, y) \in T^*\mathbb{R}^{n+1} : f(x) = 0,\ y \cdot \nabla f = 0\right\}.$$

Here, we identify T^*M to TM using the metric on M. Let $\tilde{H} = \frac{1}{2}\left(||y||^2 - 1\right)$ on $T^*\mathbb{R}^{n+1}$, and $H = \tilde{H}|_{T^*M}$. From Proposition 2.3.13, we can see that H defines the geodesic flow on T^*M with respect to the symplectic form $\omega|_{T^*M} = dy \wedge dx|_{T^*M}$.

Define $\tilde{f}, g : T^*\mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\tilde{f}(x, y) = f(x), \quad g(x, y) = y \cdot \nabla f$$

so that T^*M is the intersection $\tilde{f}^{-1}(0) \cap g^{-1}(0)$. Through straightforward computation, we find

$$X_{\tilde{H}} = y \cdot \partial_x, \quad X_{\tilde{f}} = -\nabla f \cdot \partial_y, \quad X_g = \nabla f \cdot \partial_x - \sum_{i,j} y_j f_{ij} \frac{\partial}{\partial y_i}$$

where $\partial_{x_i} f = f_i$ and $\partial_{x_i}\partial_{x_j} f = f_{ij}$. Using the formula for the Poisson bracket from Lemma 2.3.6, we obtain

$$\begin{aligned}
\{\tilde{f}, g\} &= \sum \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} = \sum \left(\frac{\partial f}{\partial x_j}\right)^2 = ||\nabla f||^2, \\
\{\tilde{f}, \tilde{H}\} &= \sum \frac{\partial f}{\partial x_j}\frac{\partial \tilde{H}}{\partial y_j} = \sum \frac{\partial f}{\partial x_j} y_j = y \cdot \nabla f = 0, \\
\{g, \tilde{H}\} &= \sum \frac{\partial g}{\partial x_j}\frac{\partial \tilde{H}}{\partial y_j} = \sum_j \frac{\partial}{\partial x_j}\left(\sum_i y_i \frac{\partial f}{\partial x_i}\right) y_j \\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} y_i y_j = \mathrm{Hess}(f)(y, y).
\end{aligned}$$

To summarize, using Proposition 2.3.8, we have the following.

Theorem 3.2.2. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$, 0 be a regular value of f, and $M = f^{-1}(0)$. Then, the geodesic vector field on T^*M is given by*

$$X_H = y \cdot \partial_x - \frac{\mathrm{Hess}(f)_x(y, y)}{\|\nabla f(x)\|^2} \nabla f \cdot \partial_y.$$

This formula can alternatively be derived by orthogonal projection.

3.2.2 Locally Symmetric Convex Hypersurfaces

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a function, and let $M = f^{-1}(0)$ be a regular level set. Suppose $N \subset M$ be a codimension 1 submanifold. We say M is a **locally symmetric convex hypersurface with fixed locus** N if there exists a reflection $\rho : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that preserves a hyperplane Σ such that

1. The hypersurface N is contained in Σ. In other words, $\rho|_N = \mathrm{Id}|_N$.
2. The reflection ρ can be regarded as a map defined on a tubular neighborhood of N. That is, there exists a tubular neighborhood $\nu(N)$ of N in M such that $\rho(\nu(N)) = \nu(N)$.
3. (Convexity) The hypersurface M has positive sectional curvature.

For convenience, we assume that the reflection is given by $\rho(x_0, \vec{x}) = (-x_0, \vec{x})$.

We now introduce equivalent conditions of f that correspond M being a locally symmetric convex hypersurface. The equivalence of these conditions will be verified in the remainder of this subsection.

(B1) For any $(x_0, \vec{x}) \in M$ with $|x_0|$ small enough, $f(x_0, \vec{x}) = f(-x_0, \vec{x})$.

(B2) The Hessian of f is positive definite.

First, assume (B1). Define

$$N = M \cap \{x \in M \,:\, x_0 = 0\}\,, \quad \nu(N) = \{x \in M \,:\, |x_0| < \varepsilon\}\,,$$

where $|x_0| < \varepsilon$ satisfies (B1). Then we can see that M is locally symmetric with fixed locus N under the reflection $\rho(x_0, \vec{x}) = (-x_0, \vec{x})$. The converse is straightforward, so we can use (B1) as an equivalent condition.

Now consider the geometry of the cotangent bundle. We denote $Y = ST^*M$ and $B = ST^*N$. Note that B can be expressed as

$$B = \{(x, y) \in ST^*M : x_0 = 0, y_0 = 0\} = ST^*N.$$

Since the reflection of Euclidean space is an isomtery, we deduce from Theorem 2.1.11 that N is a totally geodesic submanifold. Consequently, B is tangent to the geodesic vector field. Furthermore, the normal bundle of B is trivial, as shown in the following lemma.

Lemma 3.2.3. *Let M be a Riemannian manifold diffeomorphic to an n-sphere, and let $N \subset M$ be a codimension 1 closed submanifold. Then ST^*N has a trivial normal bundle in ST^*M.*

Proof. We first show that the normal bundle $\nu(N)$ of N is trivial, which is equivalent to the orientability of N. Suppose N is not orientable. Then, for any section s of $\nu(N)$ which meets the zero section transversely, there exists a loop γ such that $s|_\gamma$ has an odd number of zeros. Now assume, for contradiction, that no such γ exists. Then, for any loop $\gamma : S^1 \to M$, every transversal section of $\gamma^*\nu(N)$ has an even number of zeros. Let $w_1(\nu(N))$ be the first Stiefel-Whitney class of the normal bundle. (see [MS74] for details).

Then,

$$\langle \gamma^* w_1(\nu(N)), [S^1] \rangle = \langle w_1(\nu(N)), \gamma_*[S^1] \rangle = 0$$

for any loop γ. This implies $\langle w_1(\nu(N)), \alpha \rangle = 0$ for any $\alpha \in H_1(M; \mathbb{Z}_2)$, so $w_1(\nu(N)) = 0$. Since the first Stiefel-Whitney class classifies real line bundles, $\nu(N)$ is orientable, a contradiction.

It follows that the intersection form $[N] \cdot [\gamma]$ in $\mathbb{Z}_2$-coordinate is not 0, which is a contradiction because $H_{n-1}(S^n; \mathbb{Z}_2) = H_1(S^n; \mathbb{Z}_2) = 0$. Hence we conclude that N is orientable.

Now, let $\nu_M(N) \simeq N \times (-\epsilon, \epsilon) \subset M$. For $x \in N$, we have $T_x M = \mathbb{R} \oplus T_x N$, implying

$$ST_x M = \left\{ (t, v) \in \mathbb{R} \oplus T_x N : t^2 + \|v\|^2 = 1 \right\},$$

where $\|\cdot\|$ is the metric inherited from M. Thus, the normal fiber at $p = (x, v) \in ST^*N$ is

$$\nu_{ST^*M}(ST^*N)_p = \nu_{ST^*_x M}(ST^*_x N)_v \oplus \nu_M(N)_x \simeq \nu_{S^{n-1}}(S^n)_v \oplus \nu_M(N)_x,$$

where S^{n-1} is embedded in S^n along the equator. □

Condition (B2) implies that f is a convex function, and M bounds a compact convex domain. By Lemma 2.1.19, M is diffeomorphic to the n-sphere. To apply Corollary 2.1.17 to our hypersurface M, we compute the second fundamental form of M. The unit normal vector ν of M in $\mathbb{R}^{n+1}$ is $\nabla f / ||\nabla f||$, and we have

$$\frac{\partial}{\partial x_i} \nu = \frac{\partial}{\partial x_i} \frac{\nabla f}{||\nabla f||} = \sum_j \left(\frac{f_{ij}}{||\nabla f||} - \frac{\sum_k f_k f_{ki} f_j}{||\nabla f||^3} \right) \frac{\partial}{\partial x_j}.$$

It follows that

$$g(S(v), w) = \frac{\sum_{i,j} v_i f_{ij} w_i}{||\nabla f||} - \frac{\sum_{i,j,k} v_i f_{ik} f_k f_j w_j}{||\nabla f||^3} = \frac{\mathrm{Hess}(f)(v, w)}{||\nabla f||}$$

for $v, w \in TM$. The second term vanishes because $w \in TM$ implies that $w \cdot \nabla f = \sum_j w_j f_j = 0$. Using Corollary 2.1.17, we get the following.

Proposition 3.2.4. *Let $M = f^{-1}(0) \subset \mathbb{R}^{n+1}$ be a regular level set, and let v, w be orthogonal unit tangent vectors at a point $x \in M$. Then,*

$$K_M(v,w) = \frac{\mathrm{Hess}(f)_x(v,v)\mathrm{Hess}(f)_x(w,w) - \mathrm{Hess}(f)_x(v,w)^2}{||\nabla f(x)||^2}.$$

Proposition 3.2.5. *Let $M = f^{-1}(0) \subset \mathbb{R}^{n+1}$ be a regular level set. The sectional curvature $K_M(\sigma)$ is always positive if and only if* $\mathrm{Hess}(f)$ *is either positive definite or negative definite on M.*

Proof. Since $\mathrm{Hess}(f)_x$ is a symmetric bilinear form, there exists an orthonormal basis $\mathfrak{B}$ of T_xM that diagonalizes $\mathrm{Hess}(f)_x$. Let

$$[\mathrm{Hess}(f)_x]_{\mathfrak{B}} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

If $K_M > 0$, then $\lambda_i\lambda_j > 0$ for any i, j implying that all λ_i have the same sign. Thus, $\mathrm{Hess}(f)_x$ is either positive definite or negative definite. Since this process depends on x continuously, the sign of λ_i cannot change as $\lambda_i \neq 0$. The converse follows directly from Proposition 3.2.4, with a fact that the definiteness of $\mathrm{Hess}(f)$ is determined by the definiteness of its (2×2)-minors. □

The following corollary can also be found in various literature, for example [Sac60].

Corollary 3.2.6. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $M = f^{-1}(0) \subset \mathbb{R}^{n+1}$ be a regular hypersurface. Assume that M has a positive sectional curvature. Then M is diffeomorphic to the n-sphere.*

Proof. By Proposition 3.2.5, $\mathrm{Hess}(f)$ is either positive definite or negative definite. If $\mathrm{Hess}(f)$ is positive definite, $f^{-1}(-\infty, 0]$ is convex, and the result follows. If $\mathrm{Hess}(f)$ is negative definite, consider $\bar{f} = -f$, where $M = \bar{f}^{-1}(0)$ and $\mathrm{Hess}(\bar{f})$ is positive definite. The same argument applies. □

Thus, the condition (B2) implies convexity of M. Moreover, the case where $\mathrm{Hess}(f)$ is negative definite can be treated by using considering $-f$ instead of f. Hence, without loss of generality, we may assume (B2) to prove Theorem 3.2.1.

3.2.3 Proof of the Existence

Using the notations from Section 3.2.1 and Section 3.2.2, we assume (B1) and (B2). We define a map $\pi : Y \setminus B \to S^1 \subset \mathbb{C}$ by

$$\pi(x, y) = \frac{x_0 + iy_0}{|x_0 + iy_0|}.$$

As in Section 3.1.2, the angular form is defined by

$$\Theta = i \cdot d \log \pi = \frac{y_0 dx_0 - x_0 dy_0}{x_0^2 + y_0^2} = \frac{\theta}{x_0^2 + y_0^2}.$$

Substituting X_H, as computed in Theorem 3.2.2, into θ, we obtain the following.

$$\theta(X_H) = y_0^2 + x_0^2 \frac{\text{Hess}(f)_x(y, y)}{\|\nabla f(x)\|^2} \frac{f_0(x)}{x_0}.$$

Lemma 3.2.7. *Under the assumptions (B1) and (B2), there exists $\varepsilon > 0$, depending only on the function f, such that $\Theta(X_H) > \varepsilon$ on $Y \setminus B$.*

Proof. Define a function $A = A(x, y)$ by

$$A(x, y) = \frac{\text{Hess}(f)_x(y, y)}{\|\nabla f(x)\|^2} \frac{f_0(x)}{x_0},$$

so that $\theta(X_H) = A(x, y)x_0^2 + y_0^2$. It suffices to show that $A(x, y) > \varepsilon$ for all $(x, y) \in Y \setminus B$, since then

$$\Theta(X_H) = \frac{A(x, y)x_0^2 + y_0^2}{x_0^2 + y_0^2} > \frac{\varepsilon x_0^2 + y_0^2}{x_0^2 + y_0^2} > \varepsilon.$$

Since 0 is a regular value of f and Y is compact, there exists some $C > 0$ such that $0 < ||\nabla f||^2 \leq C$ on Y. By the compactness of Y, condition (B2) implies that there exists some $\delta > 0$ such that $\text{Hess}(f)_x(y, y) > \delta$ for all $(x, y) \in Y$.

Using condition (B1), we know that $f_0|_{x_0=0} = 0$. Thus, a Taylor expansion with respect to x_0 gives

$$f(x_0, \vec{x}) = f(0, \vec{x}) + \frac{1}{2} f_{00}(0, \vec{x}) x_0^2 + O(x_0^3),$$
$$\frac{f_0(x_0, \vec{x})}{x_0} = f_{00}(0, \vec{x}) + O(x_0).$$

Since $\mathrm{Hess}(f)_x(y, y) > \delta$, we have $f_{00}(x) = \mathrm{Hess}(f)_x((y_0, 0), (y_0, 0)) > \delta$ for all x. Choosing a small $\eta > 0$, we see that for $|x_0| < \eta$, $f_0/x_0 > \delta/2$. Consequently, $A(x, y) > \delta^2/2C$ for $|x_0| < \eta$.

For $|x_0| \geq \eta$, we only need to bound f_0/x_0. Since $f_0(0, \vec{x}) = 0$ and $f_{00} > \delta$, it follows that $f_0(x_0, \vec{x}) > 0$ when $x_0 > 0$ and $f_0(x_0, \vec{x}) < 0$ when $x_0 < 0$. By the compactness of $Y \cap \{|x_0| \geq \eta\}$, there exists some $\delta_1 > 0$ such that $f_0/x_0 > \delta_1$. This implies $A(x, y) > \delta\delta_1/C$ for $|x_0| \geq \eta$. Letting $\varepsilon = \min(\delta^2/2C, \delta\delta_1/C)$, we obtain the desired lower bound. □

Theorem 3.2.8. *Under the assumptions (B1) and (B2), the map $\pi : Y \setminus B \to S^1$ defines an open book decomposition of Y, to which the geodesic vector field is adapted.*

Proof. We will apply Lemma 2.4.4. From Lemma 3.2.7, there exists $\varepsilon > 0$ such that $\Theta(X_H) > \varepsilon$. Since $\Theta = i \cdot d \log \pi$, the first condition of Lemma 2.4.4 is satisfied.

Next, consider the trivial tubular neighborhood of B, whose existence is guaranteed by Lemma 3.2.3. Denote this tubular neighborhood by

$$\nu(B) \simeq B \times D^2$$
$$(x, y) \mapsto (\vec{x}, \vec{y}; x_0, y_0)$$

where x_0, y_0 are small enough. Note that $\pi(b, r, \theta) = e^{i\theta}$.

Since N is a totally geodesic submanifold by (B1), the geodesic vector field X_H is tangent to $ST^*N = B$. Therefore, all the assumptions of Lemma 2.4.4 are all satisfied, yielding the desired result. □

Now we can prove the first part of Theorem 3.2.1, which is the existence of a global hypersurface of section.

Theorem 3.2.9. *Under the assumptions (B1) and (B2), the geodesic flow on Y admits a global hypersurface of section, given by*

$$P = \{(x, y) \in Y : x_0 = 0, y_0 \geq 0\}.$$

Proof. As in Theorem 3.1.6, the lower bound of the angular form provided by Lemma 3.2.7 ensures the boundedness of the return time. □

We now examine the topology of the global hypersurfaces of section we have constructed.

Lemma 3.2.10. *Under the assumptions (B1) and (B2), N is diffeomorphic to S^{n-1}.*

Proof. We can express N as

$$N = \left\{(0, \vec{x}) \in \mathbb{R}^{n+1} : f(0, \vec{x}) = 0\right\} \subset \left\{(0, \vec{x}) \in \mathbb{R}^{n+1} : \vec{x} \in \mathbb{R}^n\right\} \simeq \mathbb{R}^n.$$

The convexity of f is preserved when f is restricted to the subspace $\{(0, \vec{x}) : \vec{x} \in \mathbb{R}^n\}$, which implies the result. □

Proposition 3.2.11. *The global hypersurface of section P constructed in Theorem 3.2.1 is diffeomorphic to $T^*_{\leq 1}S^{n-1}$, which is the subset of T^*S^{n-1} consisting of covectors of length ≤ 1. The boundary $B = \partial P$ is homeomorphic to ST^*S^{n-1}.*

Proof. Let p be a diffeomorphism from the upper hemisphere $H^n \subset S^n$ to the closed disk D^n. Define a map $\phi : P \to T^*_{\leq 1}N$ by $\phi(x, y) = (x, p(y))$. It is clear that ϕ is a diffeomorphism. Using Lemma 3.2.10, we conclude the result. □

3.2.4 Extension of Return Map

To complete the proof of Theorem 3.2.1, we use the computational results in this section to extend the return map to the boundary. We can take $\partial_{y_0}, \partial_{x_0}$ as a symplectic normal frame of B in Y. To compute the normal Hessian, we analyze the linearized flow of X_H along B. With the expression

$$\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{Hess}(f)(y,y)}{||\nabla f||^2} \nabla f \\ y \end{pmatrix},$$

the linearized matrix L is given by

$$L = \begin{pmatrix} 0 & -\frac{\mathrm{Hess}(f)(y,y)}{||\nabla f||^2} \mathrm{Hess}(f) \\ \mathrm{Id} & 0 \end{pmatrix}$$

along B, since the odd-differentials of f vanish along B. In particular, under the $(\partial_{y_0}, \partial_{x_0})$ frame, we have

$$L = \begin{pmatrix} 0 & -\frac{\mathrm{Hess}(f)(y,y)}{||\nabla f||^2} f_{00} \\ 1 & 0 \end{pmatrix}.$$

As in Section 3.1.3, we have $S_N = -JL$, so

$$S_N = \mathrm{diag}\left(1, \frac{\mathrm{Hess}(f)_x(y,y)}{||\nabla f(x)||^2} f_{00}(x)\right).$$

The assumption (B2) implies that Hess(f) is positive definite and f_{00} is always positive, so S_N is positive definite. Note that even if we assume Hess(f) to be negative definite, S_N remains positive definite.

Proposition 3.2.12. *Under the assumption (B1) and (B2), the return map $\Psi : \mathring{P} \to \mathring{P}$ extends to the boundary smoothly.*

Proof. This follows from Proposition 2.4.7. □

Remark 3.2.13. In [CL24], we used the fact that

$$\Theta(X_H) = \frac{Z_N^t S_N Z_N}{Z_N^t Z_N} + O(1)$$

where Z_N is the normal vector, to compute S_N.

3.2.5 Examples

Hypersurface of Revolution

Now we apply Theorem 3.2.1 to some examples and compute the return map. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy (B1) globally, (B2), and the following third condition:

(B3) if $||\vec{x}|| = ||\vec{x}\,'||$, then $f(x_0, \vec{x}) = f(x_0, \vec{x}\,')$.

For convenience, we assume that $f(0, \vec{x}) = 0$ if and only if $||\vec{x}|| = \alpha > 0$. We call a regular level set $M = f^{-1}(0)$, which satisfies (B1), (B2) and (B3), a **hypersurface of revolution**. This is a higher dimensional analogue of a surface of revolution in $\mathbb{R}^3$. In this case, we have the parametrization

$$M \cap \{(x_0, x_1, 0, \cdots, 0)\} = \{(a(\phi), \alpha\cos\phi, 0, \cdots, 0) : \phi \in \mathbb{R}\}$$

where a is a function of ϕ. For example, in the case of the ellipsoid, $a(\phi) = a_0 \sin\phi$. We apply Theorem 3.2.1 to $ST^*M = Y$, obtaining the global hypersurface of section

$$P = \{(0, \vec{x}; y_0, \vec{y}) : y_0 \geq 0\}.$$

Proposition 3.2.14. *The return map $\Psi : P \to P$ is given by*

$$\begin{aligned}\Psi\left((0,\vec{x}),(y_0,\vec{y})\right) = \Bigg(&\left(0, \cos G(\|\vec{y}\|)\vec{x} + \frac{\alpha}{\|\vec{y}\|}\sin G(\|\vec{y}\|)\vec{y}\right), \\ &\left(y_0, -\frac{\|\vec{y}\|}{\alpha}\sin G(\|\vec{y}\|)\vec{x} + \cos G(\|\vec{y}\|)\vec{y}\right)\Bigg)\end{aligned}$$

where

$$G(t) := t\int_0^{2\pi} \frac{\sqrt{\alpha^2(1-t^2)\sin^2\sigma + \{a'(\arcsin(\sqrt{1-t^2}\sin\sigma))\}^2}}{\alpha(1-(1-t^2)\sin^2\sigma)} d\sigma.$$

if $t \neq 0$, and $G(0) = 2\pi$.

Proof. First, consider the case $n = 2$, so M is 2-dimensional. Here, we take the parametrization

$$M = \{(a(\phi), \alpha \cos\phi \cos\lambda, \alpha \cos\phi \sin\lambda) : \phi, \lambda \in \mathbb{R}\}.$$

We interpret ϕ as a *latitude* and λ as a *longitude*. Consider the geodesic with initial position $(\phi_0 = 0, \lambda_0)$ and initial velocity $\cos\theta_0 \partial_\phi / \|\partial_\phi\| + \sin\theta_0 \partial_\lambda / \|\partial_\lambda\|$. Let $\Delta\lambda$ denote the change in λ after the first return time. This change equals four times the longitude shift where the x_0-coordinate of the curve becomes the maximum. The situation is illustrated in Figure 3.2[2].

Let $\theta = \theta(\phi)$ be the angle of the geodesic's velocity with respect to the longitude. Then we have Clairaut's integral (see [Arn89])

$$I(\phi) = \alpha \cos\phi \sin\theta(\phi),$$

where $I(0) = \alpha \sin\theta_0$ and $\theta_0 = \theta(0)$. Thus, $\sin\theta(\phi)\cos\phi = \sin\theta_0$. At the x_0-maximum point, $\phi = \pi/2 - \theta_0$, yielding

$$\Delta\lambda = 4\int_0^{\pi/2-\theta_0} \frac{d\lambda}{d\phi} d\phi = 4\int_0^{\pi/2-\theta_0} \frac{d\lambda}{ds}\frac{ds}{d\phi} d\phi,$$

where s is the arc-length parameter. From

$$ds^2 = (\alpha^2 \sin^2\phi + a'(\phi)^2) d\phi^2 + \alpha^2 \cos^2\phi \, d\lambda^2,$$

we obtain

$$\begin{aligned}\Delta\lambda =& 4\int_0^{\pi/2-\theta_0} \frac{\sin\theta(\phi)}{\alpha\cos\phi} \frac{\sqrt{\alpha^2\sin^2\phi + a'(\phi)^2}}{\cos\theta(\phi)} d\phi \\ &= 4\int_0^{\pi/2-\theta_0} \frac{\sin\theta_0 \sqrt{\alpha^2\sin^2\phi + a'(\phi)^2}}{\alpha\cos\phi\sqrt{\cos^2\phi - \sin^2\theta_0}} d\phi.\end{aligned}$$

To get the formula in the theorem, substitute ϕ to $\arcsin(\cos\theta_0 \sin\sigma)$.

In the general case, consider the initial condition of geodesic $(\mathbf{x}_0, \mathbf{y}_0)$ starting in P. The 3-dimensional linear subspace $S \subset \mathbb{R}^{n+1}$ containing 0, $\mathbf{x}_0$, $\mathbf{y}_0$, and the x_0-axis intersects M in a 2-dimensional ellipsoid containing the

[2]We thank Jinho Jeoung for providing the illustration.

geodesic. Thus, applying the 2-dimensional result yields the general formula. $\square$

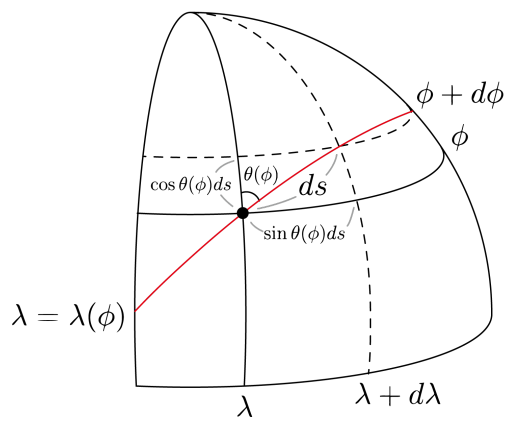

Figure 3.2: Illustration of the situation. The geodesic is drawn in red.

It is straightforward to see that the return map Ψ is a Hamiltonian diffeomorphism generated by $H(x, y) = \left(\int G\right)(||\vec{y}||)$. Some boundary behavior can be analyzed directly from the formula.

Proposition 3.2.15. *If $a'(0)/\alpha \notin \mathbb{Z}$, $\Psi|_{\partial P}$ does not have a fixed point.*

Proof. We can write $\Psi|_{\partial P}$ in matrix form,

$$\Psi(\vec{x}, \vec{y}) = A \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \cos G(1) & \alpha \sin G(1) \\ -\frac{1}{\alpha} \sin G(1) & \cos G(1) \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$$

Here, $G(1)$ is a constant independent of $\vec{x}$ or $\vec{y}$. Substituting $t = 1$ into $G(t)$, we have $G(1) = 2\pi a'(0)/\alpha$. The matrix A has a fixed point only if $\cos G(1) = 1$, or equivalently, $a'(0)/\alpha \in \mathbb{Z}$. $\square$

Ellipsoids

The result of Proposition 3.2.14 applies directly to ellipsoids with specific axis length configurations. Let $0 < a \leq 1$ and consider the function

$$f(x_0, \vec{x}) = \frac{x_0^2}{a^2} + x_1^2 + \cdots + x_n^2.$$

This is exactly the setting of Proposition 3.2.14, with $B(\phi) = a\sin\phi$, $\alpha = 1$. Then $B'(\phi) = a\cos\phi$, and

$$\begin{aligned}\{B'(\arcsin(\sqrt{1-t^2}\sin\sigma))\}^2 &= \left(-a\cos(\arcsin(\sqrt{1-t^2}\sin\sigma)\right)^2\\ &= a^2(1-(1-t^2)\sin^2\sigma).\end{aligned}$$

Therefore,

$$G(t) := t\int_0^{2\pi} \frac{\sqrt{a^2+(1-a^2)(1-t^2)\sin^2\sigma}}{1-(1-t^2)\sin^2\sigma}\,d\sigma.$$

Lemma 3.2.16. *Define* $-(1-a^2)(1-t^2)/a^2 = k$. *Then,*

$$\int \frac{\sqrt{a^2+(1-a^2)(1-t^2)\sin^2\sigma}}{1-(1-t^2)\sin^2\sigma}\,d\sigma = -\frac{1-a^2}{a}F\left(\sigma\,|k\right) + \frac{1}{a}\Pi\left(1-t^2;\sigma|k\right),$$

where F *and* Π *are the elliptic integrals of the first and third kinds, respectively:*

$$F(\phi\,|\,k) := \int_0^\phi \frac{d\theta}{\sqrt{1-k\sin^2\theta}}, \qquad \Pi(n;\phi\,|\,k) := \int_0^\phi \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k\sin^2\theta}}.$$

Proof. Let $1-a^2 = b$ and $1-t^2 = c$ for convenience. Then,

$$\begin{aligned}\int \frac{\sqrt{a^2+bc\sin^2\sigma}}{1-c\sin^2\sigma}d\sigma &= \int \frac{a^2+bc\sin^2\sigma}{(1-c\sin^2\sigma)\sqrt{a^2+bc\sin^2\sigma}}d\sigma\\ &= \int \frac{-b(1-c\sin^2\sigma)+(a^2+b)}{(1-c\sin^2\sigma)\sqrt{a^2+bc\sin^2\sigma}}d\sigma\\ &= -\frac{b}{a}\int \frac{d\sigma}{\sqrt{1+(bc/a^2)\sin^2\sigma}}\\ &\quad+\frac{1}{a}\int \frac{d\sigma}{(1-c\sin^2\sigma)\sqrt{1+(bc/a^2)\sin^2\sigma}}\\ &= -\frac{b}{a}F(\sigma|-bc/a^2) + \frac{1}{a}\Pi(c;\sigma|-bc/a^2).\end{aligned}$$

□

With Proposition 3.2.14 and Lemma 3.2.16, we can compute the return map explicitly.

Now we consider the limit cases. When $a = 1$, M is a standard sphere, and the return map converges to the identity. Setting $b = 0$, we have $F(\sigma \,|\, 0) = \sigma$. Furthermore,

$$\Pi(n; 2\pi \,|\, 0) = \int_0^{2\pi} \frac{d\theta}{1 - n\sin^2\theta} = \frac{2\pi}{\sqrt{1-n}},$$

if $0 < n < 1$, so $\Pi(1 - t^2, 2\pi \,|\, 0) = 2\pi/t$. It follows from Lemma 3.2.16 that $G(t) = 2\pi$, and τ is an identity map.

The converse limit case is $a \to 0$. In this case, the sphere deforms, and the dynamics approach a *billiard*. Define $f_a(x_0, \vec{x}) = x_0^2/a^2 + \|\vec{x}\|^2$, and $f_a^{-1}(1) = M_a$ for $0 < a \leq 1$. Let

$$M_0 = \left\{x \in \mathbb{R}^{n+1} \,:\, x_0 = 0,\, x_1^2 + \cdots + x_n^2 \leq 1\right\}.$$

We can consider the map $\beta_{ab} : M_a \to M_b$ defined by $\beta_{ab}(x_0, \vec{x}) = (bx_0/a, \vec{x})$ for $0 \leq b < a$. Then β_{ab} is a diffeomorphism for any $0 < b < a$, but β_{a0} is a 2-to-1 map outside the boundary. This means that we can consider M_0 is a 2-to-1 limit of a family $\{M_a\}$.

We also need to consider the behavior of tangent vectors on N_0. Explicitly, there is an obvious map $d\beta_{ab}$ between tangent bundles, and we know that if $b = 0$, the boundary vectors $(\pm y_0; \vec{y})$ map to the same vectors. Let

$$Y_0 = \{(\vec{x}, \vec{y}) \in T^*\mathbb{R}^n \,:\, \|\vec{x}\| \leq 1,\, \vec{y} \text{ points inward } \partial M_0 \text{ if } \|\vec{x}\| = 1,\, \|\vec{y}\| \leq 1\},$$

then this is a 2-to-1 limit of $Y_a = T^*M_a$.

It's interesting to study the limit of geodesics on M_0, which become straight lines in the interior and *reflect* at the boundary. This dynamics on Y_0 is called a **billiard**. The global hypersurfaces of section we constructed also converge to a certain hypersurface in Y_0. That is,

$$P_0 = \{(\vec{x}, \vec{y}) \in T^*\mathbb{R}^n \,:\, \|\vec{x}\| = 1,\, \vec{y} \text{ points inward to } \partial M_0,\, \|\vec{y}\| \leq 1\}.$$

Proposition 3.2.17. *The submanifold P_0 is a global hypersurface of the section of the billiard on Y_0, and the return map is given by*

$$\Psi(\vec{x},\vec{y}) = \left(\vec{x}\cos\theta_0 + \frac{\vec{y}}{\|\vec{y}\|}\sin\theta_0, \vec{y}\cos\theta_0 - \|\vec{y}\|\vec{x}\sin\theta_0\right),$$

where $\theta_0 = 2\arccos\|\vec{y}\|$.

Proof. A billiard trajectory lies on a 2-dimensional subspace that contains the point $\vec{x}$, direction $\vec{y}$, and the origin. Thus, the problem reduces to finding the return map for a 2-dimensional billiard. After an appropriate rotation, we can map $(0,\vec{x})$ to e_1 and $(0,\vec{y}/||\vec{y}||)$ to e_2 in $\mathbb{R}^2$. The situation is illustrated in the Figure 3.3[3]. Notice that the length of $(y_0,\vec{y})$, the red vector in Figure 3.3, is 1. Thus, the problem of finding the return map of the billiard simplifies to an elementary geometric problem. □

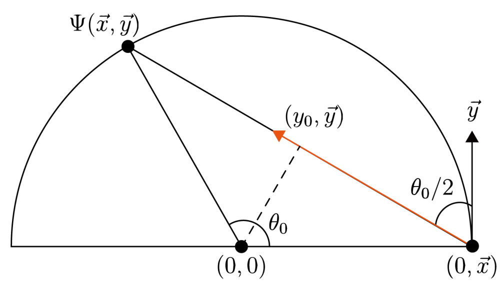

Figure 3.3: Return map of the billiard in dimension 2.

Returning to Proposition 3.2.14 and Lemma 3.2.16, setting $a = 0$ in $G(t)$, we have

$$\begin{aligned} G(t) &= \int_0^{2\pi} \frac{t\sqrt{(1-t^2)\sin^2\sigma}}{1-(1-t^2)\sin^2\sigma}\, d\sigma \\ &= -\frac{|\sin\sigma|}{\sin\sigma}\cdot\left(\arctan\frac{\sqrt{1-t^2}-\tan\sigma}{t} + \arctan\frac{\sqrt{1-t^2}+\tan\sigma}{t}\right)\Bigg|_0^{2\pi} \\ &= 4\arctan\frac{\sqrt{1-t^2}}{t} = 4\arccos t. \end{aligned}$$

[3]Again, we thank Jinho Jeoung for the nice illustration.

Substituting this into Proposition 3.2.14, we see that this is exactly twice the return map in Proposition 3.2.17. The reason the two return maps are not the same but twice as large is as follows: Consider a point $(x, y) = (0, \vec{x}; y_0, \vec{y})$ in P_a with $y_0 > 0$. Following the geodesic with initial condition (x, y), there exists a minimal $t_0 > 0$ such that $\gamma(t_0) = (0, \vec{x}'; y_0', \vec{y}')$. However, we must have $y_0' < 0$ in this case. In fact, this point corresponds to where the angular form becomes exactly π. This point also converges to the point in P_0 in limit, since it's a 2-to-1 limit. Hence, the limit of the return map of P_a does not converge to the first return map, but to the *second return map* of billiard.

Kepler Problem

Consider the Kepler Hamiltonian defined on $T^*\mathbb{R}^3 \setminus \{0\}$,

$$E(q, p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

The Hamiltonian is singular at the origin, but we can apply Moser regularization, which will be discussed in Section 4.1.3, to regularize each level set. The regularization for the level set with Kepler energy E_0 is given by the composition of the switch map and the stereographic projection from the unit cotangent bundle of S^3 with radius $r = \sqrt{-2E_0}$. The regularized system with Kepler energy E_0 can be regarded as a geodesic flow on the unit cotangent bundle of $S^3 \subset \mathbb{R}^4$ with radius $r = \sqrt{-2E_0}$. The global hypersurface of section is given by

$$P = \{(x, y) \,:\, x_3 = 0,\, y_3 \geq 0\}.$$

In $T^*\mathbb{R}^3$, we have

$$\begin{aligned} x_3 &= \frac{2r^2 p_3}{p^2 + r^2} = 0 \\ y_3 &= \frac{p^2 + r^2}{2r^2} q_3 - \frac{p \cdot q}{r^2} p_3 \geq 0. \end{aligned}$$

Hence,

$$P = \left\{(q, p) \in E^{-1}(c) \,:\, p_3 = 0,\, q_3 \geq 0\right\}.$$

This is the set of points of maximum height of the Kepler orbits, and the Reeb dynamics on the binding correspond to the planar Kepler problem.

Chapter 4

Periodic Orbits of Spatial Kepler Problem

In this chapter, we investigate the periodic orbits of the spatial Kepler problem. We review the generalities of the Kepler problem in Section 4.1, and the planar rotating Kepler problem in Section 4.2. In Section 4.3, we examine the moduli space of the periodic orbits of the spatial Kepler problem and classify all periodic orbits of the spatial rotating Kepler problem. In Section 4.4, we compute the Conley-Zehnder index of the periodic orbits of the spatial Kepler problem and relate it to the symplectic homology of the cotangent bundle of the 3-sphere. This chapter is based on joint work with Beomjun Sohn.

4.1 Kepler Problem

A **Kepler problem** is a dynamical system that describes the motion of a mass-less body in Euclidean space under the gravitational force exerted by another body. For simplicity, we assume that the mass of the source of gravitational force is 1 and it lies at the origin. The Hamiltonian describing the Kepler problem is given by

$$\begin{aligned} E : T^*(\mathbb{R}^3 \setminus \{0\}) &\to \mathbb{R} \\ (q,p) &\mapsto \frac{1}{2}|p|^2 - \frac{1}{|q|}. \end{aligned}$$

Here, q is the position, which is the point on the base manifold $\mathbb{R}^3 \setminus \{0\}$, and p is the momentum, which is the vector in $T_q(\mathbb{R}^3 \setminus \{0\})$. The phase space $T^*\mathbb{R}^3$ is equipped with the natural symplectic form $dp \wedge dq = \sum dp_i \wedge dq_i$. The Hamiltonian E describes the mechanical energy of the mass-less body. Solving the Hamiltonian equation for the Kepler problem is straightforward; we have

$$X_E = p \cdot \partial_q - \frac{q}{|q|^3} \cdot \partial_p$$

or equivalently

$$\dot{p} = -\frac{q}{|q|^3}, \qquad \dot{q} = p.$$

4.1.1 Invariants of Kepler Problem

Before we investigate the solutions of the Kepler problem, we introduce two important invariants associated with it.

The Kepler problem has an obvious $SO(3)$-symmetry. By Theorem 2.3.11, there exists an invariant corresponding to this symmetry. This invariant is called the **angular momentum** and is defined by

$$L = (L_1, L_2, L_3) = q \times p$$

where $\times$ denotes the cross product. To verify this, consider L_3, which corresponds to rotation along the q_3- and p_3-axis. The corresponding Hamiltonian vector field is given by

$$X_{L_3} = -q_2 \partial_{q_1} + q_1 \partial_{q_2} - p_2 \partial_{p_1} + p_1 \partial_{p_2}.$$

By integrating, we find that X_{L_3} generates a rotation around the origin,

$$Fl_t^{X_{L_3}}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q_1\cos t - q_2\sin t & q_1\sin t + q_2\cos t & q_3 \\ p_1\cos t - p_2\sin t & p_1\sin t + p_2\cos t & p_3 \end{pmatrix}.$$

Thus, we conclude that each component of L in $T^*\mathbb{R}^3$ corresponds to the rotational symmetry along each axis, and $\{E, L_i\} = 0$ for each $i = 1, 2, 3$.

We now introduce another invariant of the Kepler problem: the **Laplace-Runge-Lenz vector**, defined as

$$A = p \times L - \frac{q}{|q|}.$$

It is clear that A is perpendicular to L.

Lemma 4.1.1. *The Laplace-Runge-Lenz vector A is an invariant of the Kepler problem. In other words, $\{E, A\} = 0$, or equivalently, A is constant along the Kepler orbit.*

Proof. We compute the time derivative of A along the Kepler orbit,

$$\begin{aligned} \dot{A} &= \dot{p} \times L + p \times \dot{L} - \frac{\dot{q}}{|q|} + \frac{(q \cdot \dot{q})q}{|q|^3} \\ &= \frac{1}{|q|^3}(-q \times (q \times p) - |q|^2 p + (q \cdot p)q) = 0. \end{aligned}$$

In the second equality, we used the Hamiltonian equation $\dot{p} = -q/|q|^3$ and $\dot{q} = p$, as well as the invariance of angular momentum $\dot{L} = 0$. The last equality follows from an elementary identity involving the cross product. □

Remark 4.1.2. From the perspective of Theorem 2.3.11, there must be a symmetry corresponding to the invariant A. However, unlike the case of angular momentum, this symmetry is not directly observable. The Kepler problem can be described as a geodesic flow on S^3 via Moser regularization, as described in Section 4.1.3. The geodesic flow on S^3 possesses an $SO(4)$-symmetry, which is 6 dimensional. The symmetry corresponding to A originates from this structure and is often referred to as the *hidden symmetry* of the Laplace-Runge-Lenz vector.

4.1.2 Three Laws of Kepler

We begin by stating the celebrated three laws of Kepler.

Theorem 4.1.3 (Kepler's Law)**.** *The following hold for the orbits of the Kepler problem.*

1. *The orbits are conic sections with one focus at the origin. If the energy is negative, the orbits are ellipses.*
2. *In polar coordinates on the plane contains an orbit, the* ***areal velocity*** $\dot{S} = r^2\dot{\theta}/2$ *is constant.*
3. *If the orbit is closed, the period* τ *of the orbit is given by*

$$\tau^2 = -\frac{\pi^2}{2E^3}.$$

We now prove the first law in Theorem 4.1.3. This proof follows the approach in [FvK18].

Theorem 4.1.4. *The orbits of the Kepler problem are conic sections, with eccentricity* ε *satisfying*

$$\varepsilon^2 = 2E|L|^2 + 1.$$

The equation of the orbit in polar coordinates on the plane $L \cdot q = 0$ *is given by*

$$r = \frac{|L|^2}{1 + |A|\cos(\theta - g)}.$$

Proof. We first observe that $\langle A, L\rangle = 0$, and compute

$$|L|^2 = \langle q \times p, L\rangle = \langle p \times L, q\rangle = \left\langle A + \frac{q}{|q|}, q \right\rangle = \langle A, q\rangle + |q|$$

After rotating the coordinate system to align L with the q_3-axis, we can say $L = (0, 0, l)$ and $A = (|A|\cos g, |A|\sin g, 0)$. Write $q = (r\cos\theta, r\sin\theta, 0)$. Substituting these into the equation, we find

$$r = \frac{l^2}{1 + |A|\cos(\theta - g)}.$$

This is the polar equation of a conic section, with eccentricity equal to $|A|$.

Finally, we prove that $|A|^2 = 2E|L|^2 + 1$. From direct computation,

$$\begin{aligned} |A|^2 &= |p \times L|^2 - \frac{2}{|q|}\langle p \times L, q\rangle + 1 = |p|^2|L|^2 - \frac{2}{|q|}\langle q \times p, L\rangle + 1 \\ &= |p|^2|L|^2 - \frac{2}{|q|}|L|^2 + 1 = 2\left(|p|^2 - \frac{1}{|q|}\right)|L|^2 + 1. \end{aligned}$$

□

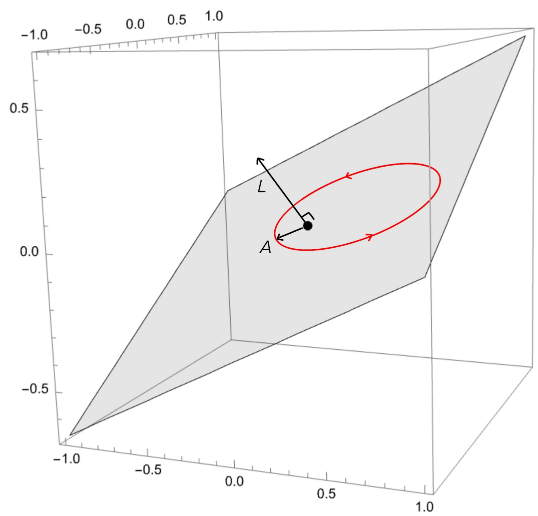

Figure 4.1: Illustration of the Kepler orbit determined by L and A.

Corollary 4.1.5. *The Laplace-Runge-Lenz vector is parallel to the major axis of the Kepler orbit.*

Proof. The formula in the proof of the previous theorem implies that q aligns with the major axis when $\theta = g$, at which point q is parallel to A. In particular, A points toward the closest point on the major axis to the origin. This is illustrated in Figure 4.2. □

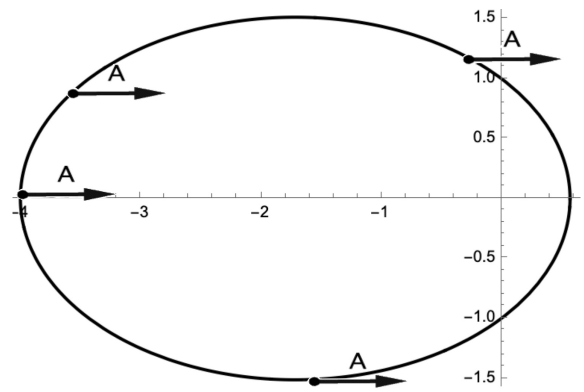

Figure 4.2: Illustration of the Laplace-Runge-Lenz vector for $L = (0, 0, 1)$ and $A = (\sqrt{3}/2, 0, 0)$.

Due to the $SO(3)$-symmetry, it suffices to consider orbits lying in $\mathbb{R}^2$ or the $q_1 q_2$-plane. Using polar coordinates on $\mathbb{R}^2$, we write

$$(q_1, q_2) = (r\cos\theta, r\sin\theta).$$

To ensure the coordinate change is symplectic, the momenta transform as

$$p_1 dq_1 + p_2 dq_2 = p_r dr + p_\theta d\theta$$

yielding

$$(p_1, p_2) = \left(\cos\theta p_r - \frac{\sin\theta}{r} p_\theta, \sin\theta p_r + \frac{\cos\theta}{r} p_\theta\right).$$

In this representation, the energy E and angular momentum L are expressed as

$$E(r, \theta) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{1}{r},$$
$$L = r\cos\theta p_2 - r\sin\theta p_1 = p_\theta$$

From the Hamiltonian equation $\dot{\theta} = \partial_{p_\theta} E$ implies

$$\dot{\theta} = \frac{p_\theta}{r^2} = \frac{L}{r^2}.$$

Theorem 4.1.6. *The areal velocity of the orbit of a Kepler problem is con-*

stant.

Proof. Let γ be a Kepler orbit with an initial point on the q_1-axis. Denote the area swept out by γ from time 0 to t by $\mathcal{A}(t)$. We have $d\mathcal{A}/d\theta = r^2/2$,, so

$$\mathcal{A}(t) = \frac{1}{2}\int_0^t r^2 d\theta = \frac{1}{2}\int_0^t r^2 \dot{\theta} dt = \frac{1}{2}\int_0^t L dt = \frac{Lt}{2}$$

Thus, $\dot{\mathcal{A}}(t) = L/2$, and since L is invariant, the areal velocity is constant. □

Theorem 4.1.7. *Let τ be the period of Kepler orbit. Then we have*

$$\tau^2 = -\frac{\pi^2}{2E^3}.$$

Proof. From the previous theorem, we have

$$\pi ab = \mathcal{A}(\tau) = \frac{L\tau}{2}$$

where a and b are the semi-major and semi-minor axes, respectively. The eccentricity of the ellipse satisfies $\varepsilon^2 = 2EL^2 + 1$, and the semi-major axis is given by

$$a = \frac{1}{2}\left(\frac{L^2}{1+|A|} + \frac{L^2}{1-|A|}\right) = \frac{L^2}{1-|A|^2}.$$

The semi-minor axis $b = \sqrt{1-\varepsilon^2}a$, so

$$\pi ab = \pi a^2 \varepsilon = \frac{L^4 \pi \sqrt{1-|A|^2}}{(1-|A|^2)^2} = \frac{\pi L^4}{(1-|A|^2)^{3/2}}$$

Squaring the earlier equation,

$$\tau^2 = \frac{4\mathcal{A}^2}{L^2} = \frac{4\pi^2 L^8}{L^2(1-|A|^2)^3} = \frac{4\pi^2 L^8}{-8E^3 L^8} = -\frac{\pi^2}{2E^3}$$

□

4.1.3 Moser Regularization

The singularity at the origin in the Kepler problem prevents the level set from being compact or complete. To address this, various regularization methods have been developed. In this subsection, we introduce the regularization method for the Kepler problem proposed by Moser in [Mos70]. We begin by noting the formula used in the regularization process.

Lemma 4.1.8. *Let S_r^n be the n-sphere of radius r. The stereographic projection from the north pole $(r, 0, \cdots, 0)$ is given by*

$$\begin{aligned}\Phi_r : T^*S_r^3 &\to T^*\mathbb{R}^3\\ (x,y) &\mapsto \left(\frac{r\vec{x}}{r-x_0}, \frac{r-x_0}{r}\vec{y} + \frac{y_0}{r}\vec{x}\right)\end{aligned}$$

and the inverse is given by

$$\begin{aligned}\Psi_r : T^*\mathbb{R}^3 &\to T^*S_r^3\\ (p,q) &\mapsto \left(\frac{r(p^2-r^2)}{p^2+r^2}, \frac{2r^2p}{p^2+r^2}, \frac{p\cdot q}{r}, \frac{p^2+r^2}{2r^2}q - \frac{p\cdot q}{r^2}p\right)\end{aligned}$$

In particular, the following relations hold.

$$\begin{aligned} r - x_0 &= \frac{2r^3}{p^2+r^2}, & p^2 &= \frac{2r^3}{r-x_0} - r^2,\\ |y|^2 &= \frac{(p^2+r^2)^2}{4r^4}|q|^2, & |q| &= \frac{2r^2}{p^2+r^2}|y| = \frac{r-x_0}{r}|y|.\end{aligned}$$

The following procedure is known as the **Moser regularization** of the Kepler problem. We fix the energy level $E = E_0 < 0$. Consider the Hamiltonian on $T^*(\mathbb{R}^3 \setminus \{0\})$,

$$\tilde{K}_{E_0}(q,p) = \frac{1}{2}\left(|q|\left(E(q,p) - E_0\right) + 1\right)^2 = \frac{1}{2}\left(\frac{1}{2}(|p|^2 - 2E_0)|q|\right)^2.$$

The Hamiltonian $\tilde{K}_{E_0}$ extends smoothly to the origin, allowing it to be defined on $T^*\mathbb{R}^3$. Moreover, the regular level set $E^{-1}(E_0)$ is a subset of $\tilde{K}^{-1}(1/2)$, so the Hamiltonian flows of E and $\tilde{K}_{E_0}$ are parallel on this level set.

The **switch map** σ is a symplectomorphism defined on $T^*\mathbb{R}^3$ as $\sigma(q,p) = (p,-q)$. By composing $\tilde{K}_{E_0}$ with the switch map and the inverse stereographic projection $\Psi_r : T^*\mathbb{R}^3 \to T^*S^3_r$, we obtain a Hamiltonian on $T^*S^3_r$,

$$K_r(x,y) = \tilde{K}_{E_0}(\Psi_r(p,-q)) = \frac{1}{2}|y|^2 \left(r^2 + (r - x_0)\left(-\frac{E_0}{r^2} - \frac{1}{2}\right)\right)^2.$$

Setting $r = \sqrt{-2E_0}$, we simplify:

$$K_r(x,y) = \frac{r^4}{2}|y|^2.$$

The Hamiltonian K_r defines the geodesic flow on $T^*S^3_r$. In short, we can embed the level set $E = E_0$ of the Kepler problem as a sub-system of the geodesic flow on $T^*S^3_r$.

For convenience, we pull back K_r to $T^*S^3 = T^*S^3_1$ using the scaling map

$$\begin{aligned} S_{1,r} : T^*S^3 &\to T^*S^3_r \\ (x,y) &\mapsto (rx, y/r). \end{aligned}$$

The Hamiltonian on T^*S^3 then becomes

$$K_r(x,y) = \frac{r^2}{2}|y|^2.$$

For the future reference, we present the following lemma froom [MvK22a], which can also be derived from Proposition 2.3.8.

Lemma 4.1.9. *Let $T^*S^n \subset T^*\mathbb{R}^{n+1}$ be defined by equations $|x|^2 = 1$ and $\langle x, y\rangle = 0$. Let $K : T^*S^n \to \mathbb{R}$ be a Hamiltonian of a form*

$$K(x,y) = \frac{1}{2}|y|^2 f(x,y)^2.$$

Then, the Hamiltonian vector field of K is given by

$$\begin{aligned} X_K = &(f^2 y + |y|^2 f(f_y - (f_y \cdot x)x)) \cdot \partial_x \\ &+ |y|^2 f((f_y \cdot x)y - f_x - (f + f_y \cdot y - f_x \cdot x)x) \cdot \partial_y \end{aligned}$$

Applying this lemma with $f(x, y) = r^2$, we find

$$X_K = r^2 y \cdot \partial_x - r^2 |y|^2 x \cdot \partial_y.$$

Imposing the energy condition $K(x, y) = 1/2$, we have $|y|^2 = 1/r^2$. Thus, the equations of motion become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r^2 y \\ -x \end{pmatrix}$$

These orbits correspond to great circles on T^*S^3 with speed $1/r$. With this, we investigate the orbits added during the process of Moser regularization. We impose the initial condition

$$(q(0); p(0)) = \left(0, 0, -\frac{1}{E_0}\,;\, 0, 0, 0\right) = \left(0, 0, \frac{2}{r^2}\,;\, 0, 0, 0\right).$$

Intuitively, this condition describes an object starting to fall directly into the origin, eventually colliding with it in finite time. On T^*S^3, this corresponds to the initial condition

$$(x(0); y(0)) = \left(-1, 0, 0, 0\,;\, 0, 0, 0, -\frac{1}{r}\right).$$

The orbit of X_{K_r} on T^*S^3 with this initial condition is given by

$$(x(t); y(t)) = \left(-\cos(rt), 0, 0, -\sin(rt)\,;\, \frac{\sin(rt)}{r}, 0, 0, -\frac{\cos(rt)}{r}\right).$$

Applying the stereographic projection and the inverse switch map, we obtain the orbit in $T^*\mathbb{R}^3$

$$(q(t); p(t)) = \left(0, 0, \frac{1}{r^2}(1 + \cos(rt))\,;\, 0, 0, -\frac{r \sin(rt)}{1 + \cos(rt)}\right).$$

For any initial condition with $p(0) = 0$, we obtain the same type of orbit. This type of orbit is referred as a **collision orbit** and is illustrated in Figure 4.3. We also describe the behavior of this orbit in $T^*\mathbb{R}^3$.

1. At $t = 0$: q reaches its maximum height $|q| = -1/E_0$, and $p = 0$.
2. For $t \in [0, \pi/r)$: q moves toward the origin, and p diverges to $-\infty$.
3. For $t \in (\pi/r, 2\pi/r]$: q moves away from the origin, and p decreases from ∞.
4. At $t = 2\pi/r$: q returns to its maximum heights, and $p = 0$.

In summary, the collision orbit oscillates between the origin and the maximum height in $T^*\mathbb{R}^3$.

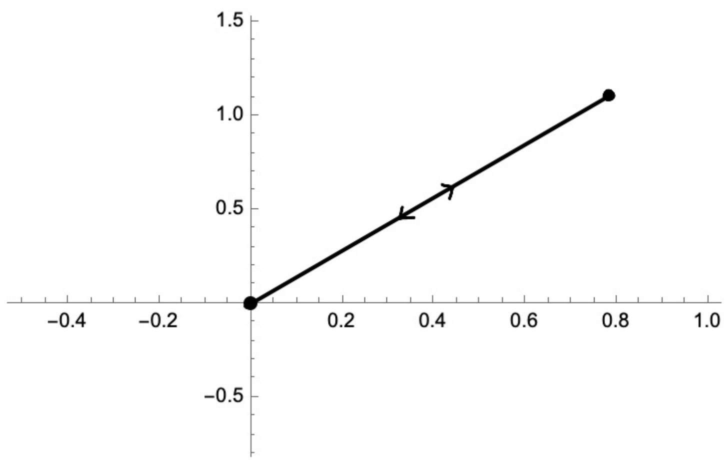

Figure 4.3: Illustration of the collision orbit.

Additionally, the above parameterization of the collision orbit γ is a reparametrization of the Kepler orbit, meaning the orbital speed differs. In particular, the period of γ as a Kepler orbit is not equal to $2\pi r$. However, γ can be regarded as a special case of an elliptical Kepler orbit with eccentricity 1. In this sense, the period of γ satisfies the Kepler's third law, which depends solely on the Kepler energy.

4.2 Planar Rotating Kepler Problem

The **planar rotating Kepler problem** is the Kepler problem restricted to $\mathbb{R}^2$ with a rotating frame around q_3- and p_3-axes. It is described by the

Hamiltonian

$$H = E + L_3 = \frac{1}{2}|p|^2 - \frac{1}{|q|} + (q_1 p_2 - q_2 p_1).$$

Here, E is referred to as the **Kepler energy**, and H as the **total energy** when clarification of the term “energy” is necessary.

Note 4.2.1. The concept of the rotating Kepler problem originates from the restricted circular three-body problem. In the circular three-body problem, the motion of two primary bodies is assumed to be circular. The Hamiltonian for this system is

$$E_t(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - m(t)|} - \frac{1-\mu}{|q - e(t)|}.$$

where μ is a mass ratio, $e(t) = -\mu(\cos t, -\sin t)$, and $m(t) = (1-\mu)(\cos t, -\sin t)$. This Hamiltonian is time-dependent. However, by adding an angular momentum term to account for the motion of the two bodies, we obtain the following autonomous Hamiltonian

$$H = \frac{1}{2}|p|^2 - \frac{\mu}{|q-(1-\mu)|} - \frac{1-\mu}{|q-\mu|} + (q_1 p_2 - q_2 p_1)$$

An autonomous Hamiltonian is easier to analyze and handle mathematically. The rotating Kepler problem is a limiting case of this system, where $\mu \to 0$.

The Hamiltonian can be reformulated as follows

$$H = \frac{1}{2}|\tilde{p}|^2 + U(|q|)$$

where $\tilde{p} = (p_1 - q_2, p_2 + q_1)$, $U(r) = -1/r - r^2/2$. The function U is called the **effective potential**, as it encapsulates all terms that depend only on q. The graph of U is shown in Figure 4.4.

Let $\pi : T^*(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}^2 \setminus \{0\}$ be the projection. The **Hill's region** for a given energy level $c < 0$ is defined as

$$\mathcal{H}_c = \{q : U(q) \leq c\}.$$

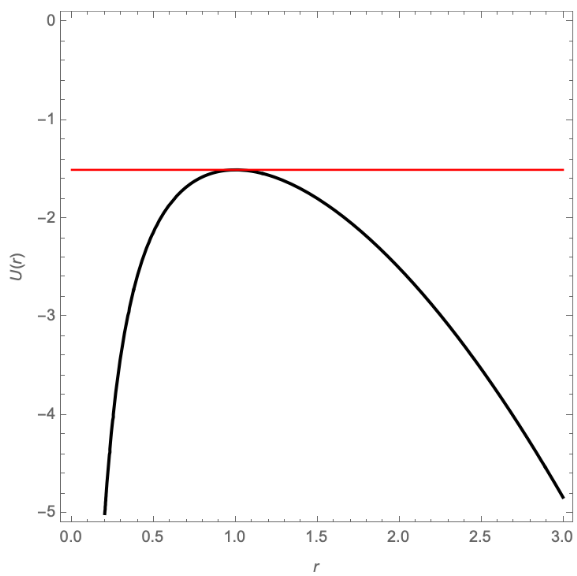

Figure 4.4: The graph of effective potential U. The maximum value is $-3/2$.

Since the term $|\tilde{p}|^2$ is always nonnegative, it follows that if $H(q,p) \leq c$, then q must belong to $\mathcal{H}_c$. The effective potential $U(r)$ has a unique maximum value of $-3/2$ at $r = 1$. It follows that when $c < -3/2$, the Hill's region $\mathcal{H}_c$ consists of one bounded component and one unbounded component. When $c < -3/2$, the Hill's region equals to $\mathbb{R}^2 \setminus \{0\}$. An illustration of the Hill's region for an energy level of -1.6 is shown in Figure 4.5. We are particularly interested in the orbits that lie within the bounded component of the Hill's region.

4.2.1 Moser Regularization

We can apply Moser regularization to the planar rotating Kepler problem, following the procedure outlined in Section 4.1.3. For an energy level $c < -3/2$, consider the Hamiltonian

$$H_\alpha(p,q) = \frac{1}{2}|p|^2 - \frac{1}{|q|} + \alpha(q_1p_2 - q_2p_1)$$

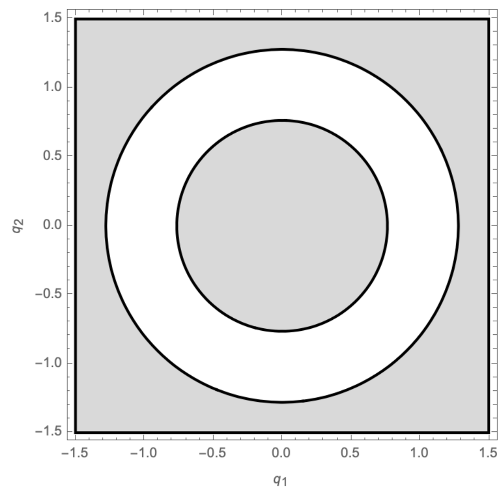

Figure 4.5: The illustration of Hill's region of energy -1.6.

where α is a parameter introduced to track the role of angular momentum. We define the following Hamiltonian on the bounded component of the Hill's region

$$K_{\alpha,c}(p,q) = \frac{1}{2}\left((H-c)|q|+1\right)^2.$$

Next, we pull back $K_{\alpha,c}$ to $T^*S^2_r$ via stereographic projection and the switch map,

$$K_{\alpha,c}(x,y) = \frac{1}{2}|y|^2\left(r^2+(r-x_0)\left(\frac{\alpha}{r^2}(x_1y_2-x_2y_1)-\frac{c}{r^2}-\frac{1}{2}\right)\right)^2.$$

Here, we take $r=\sqrt{-2c}$ and $a=\alpha/r^2$, simplifying the expression to

$$K_{\alpha,c} = \frac{1}{2}|y|^2\left(r^2+a(r-x_0)(x_1y_2-x_2y_1)\right)^2.$$

We further pull-back $K_{\alpha,c}$ to $T^*S^3_1$ using a scaling map. On T^*S^2, the Hamiltonian becomes

$$K_{a,c}(x,y) = \frac{1}{2}|y|^2\left(r+a(1-x_0)(x_1y_2-x_2y_1)\right)^2.$$

If we set $a = 0$, the Hamiltonian reduces to the Moser-regularized Kepler Hamiltonian described in Section 4.1.3.

4.2.2 Periodic Orbits

We investigate possible periodic orbits of the planar rotating Kepler problem. Since $\{E, L_3\} = 0$, we have

$$Fl_t^{X_H} = Fl_t^{X_E} \circ Fl_t^{X_{L_3}}.$$

As shown in Section 4.1, the periodic orbits of X_E are ellipses. The flow $Fl^{X_{L_3}}$ represents a rotation about the q_3- and p_3-axes with a period of 2π.

The first case is the circular orbits. We require

$$\varepsilon^2 = 2EL_3^2 + 1 = 2E(c - E)^2 + 1 = 0$$

at the energy level $H = c$. In this case, the orbit of H is automatically periodic. A straightforward computation shows that if $c < -3/2$, there are three possible values of E, and if $c > -3/2$ there is only one possible value of E.

Assume $c < -3/2$, so there are three distinct circular orbits.

1. The **retrograde orbit**, denoted by γ_+, corresponds to the smallest Kepler energy.

2. The **direct orbit**, denoted by γ_-, corresponds to the second smallest Kepler energy.

3. The orbit with the largest Kepler energy is the **outer direct orbit**, lying outside the bounded component of the Hill's region, and is not of interest here.

The retrograde and direct orbits are distinguished by the sign of angular momentum. For $\varepsilon^2 = 0$, we have $L_3 = \pm 1/\sqrt{-2E}$. Thus, the retrograde orbit (with lower Kepler energy) has positive angular momentum, while the direct orbit has negative angular momentum. From the graph, the retrograde orbit is the one that persists beyond the critical energy level $c = -3/2$.

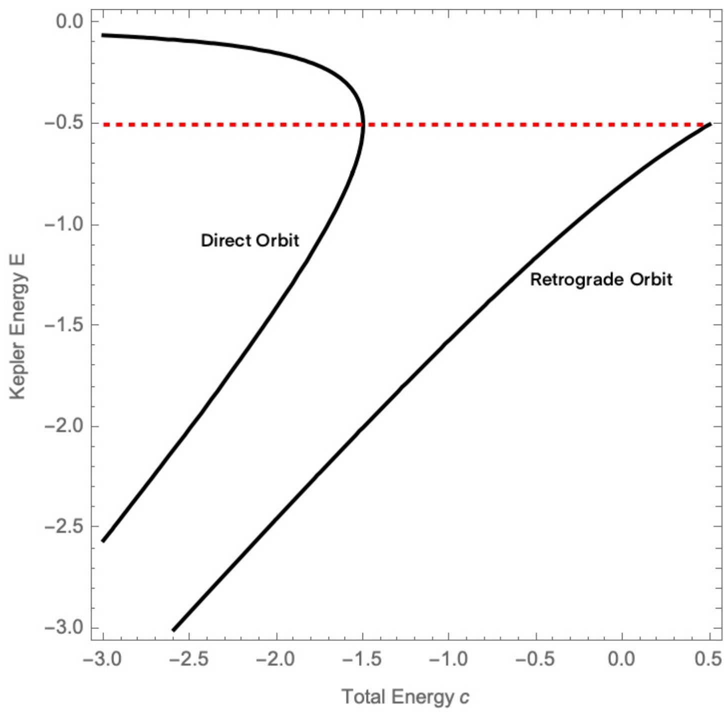

Figure 4.6: Graph of $2E(c-E)^2+1=0$. The critical energy level $E=-1/2$ corresponds to $c=-3/2$.

Proposition 4.2.2. *Let γ_+ (retrograde) and γ_- (direct) be circular orbits of the planar rotating Kepler problem with $c<-3/2$. The total energy $c^{\pm}_{k,l}$ is given by*

$$c^{\pm}_{k,l} = E_{k,l} \pm \sqrt{\frac{1}{-2E_{k,l}}}$$

while the periods of $\gamma_\pm$ is are

$$\tau_\pm = \frac{2\pi}{(-2E)^{3/2} \pm 1}.$$

Proof. For the circular orbits, $\varepsilon^2 = 2EL_3^2 + 1 = 0$, so $L_3 = \pm 1/\sqrt{-2E}$, proving the first statement.

To find the periods, we parametrize the orbits explicitly (see [AFFvK13], Section B.2). Using the transformation $(q_1, q_2) = (r\cos\theta, r\sin\theta)$ from Section 4.1, the Hamiltonian becomes

$$H(r,\theta;p_r,p_\theta) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{1}{r} + p_\theta$$

The Hamiltonian vector field is

$$X_H = \frac{p_\theta^2 - r}{r^3}\partial_{p_r} + p_r\partial_r + \left(\frac{p_\theta}{r^2} + 1\right)\partial_\theta$$

For circular orbits ($r = r_0$), $p_r = 0$ and $p_\theta^2 = r_0$. The solution is

$$\begin{pmatrix} r(t) \\ \theta(t) \\ p_r(t) \\ p_\theta(t) \end{pmatrix} = \begin{pmatrix} r_0 \\ \left(\pm\frac{1}{r_0^{3/2}} + 1\right)t \\ 0 \\ \pm\sqrt{r_0} \end{pmatrix}$$

Here, $p_\theta = \sqrt{r_0}$ corresponds to the retrograde orbit ($L_3 > 0$), while $p_\theta = -\sqrt{r_0}$ corresponds to the direct orbit.

Using $E = -1/2r_0$, substituting $r_0 = -1/2E$ into the period formula completes the proof. □

In the second case, the orbit of E is not circular. Here, the period of X_E-orbit must resonate with the 2π-periodic flow $Fl^{X_{L_3}}$. As a result, the orbit must be $2\pi(l/k)$-periodic for some $k, l \in \mathbb{N}$.

From Kepler's third law, the energy of such an orbit is given by

$$E_{k,l} = -\frac{1}{2}\left(\frac{k}{l}\right)^{3/2}.$$

An example of such an orbit is shown in Figure 4.8[1]. These orbits also possess rotational symmetry. Thus, for each pair (k,l) and $|L_3| < 1/\sqrt{-2E_{k,l}}$, there exists an S^1-family of elliptic orbits that form a torus. Additionally, there is a retrograde orbit and a direct orbit with Kepler energy $E_{k,l}$, whose total energies are denoted $c_{k,l}^{\pm}$.

[1] We appreciate Chankyu Joung for illustrating this diagram.

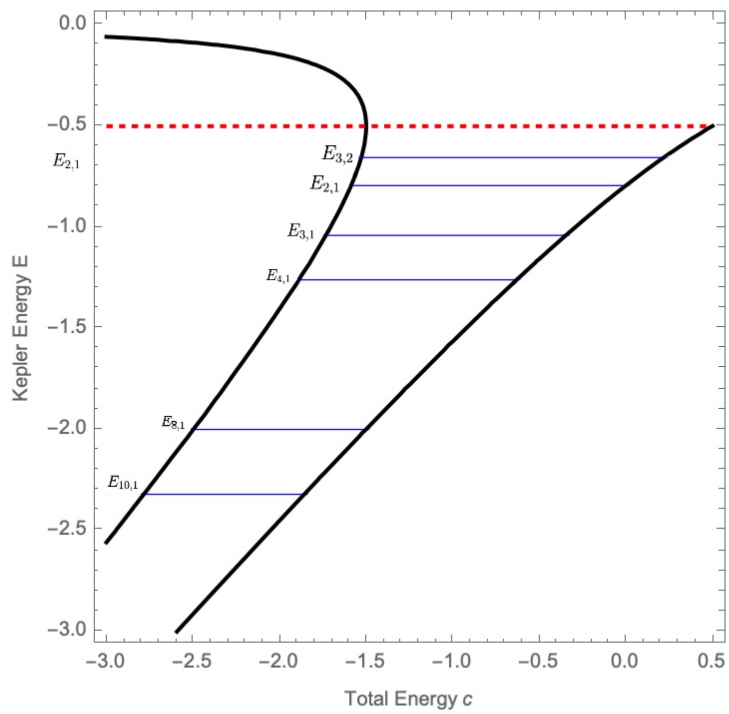

Figure 4.7: Illustration of the birth and death of elliptic orbits. Each interior point on the segments corresponds to an S^1-family of orbits.

To summarize, for given $E_{k,l}$, there exists an S^1-family of orbits for each total energy between $c^-_{k,l}$ and $c^+_{k,l}$. At the boundary energies $c^\pm_{k,l}$, these families degenerate into retrograde or direct circular orbits. This observation is a key concept used in computing the Conley-Zehnder index of closed orbits in the rotating Kepler problem, as described in [AFFvK13]. Here, we present the result without detailed derivation.

Theorem 4.2.3 ([AFFvK13])**.** *Let $\gamma^N_\pm$ denote the N-th iterate of the simple retrograde and direct orbits of rotating the Kepler problem. Assume that $N\tau_\pm \notin \mathbb{Z}\frac{2\pi}{(-2E)^{3/2}}$. Then,*

$$\mu_{CZ}(\gamma^N_\pm) = 1 + 2\max\left\{k \in \mathbb{Z} \,|\, k\frac{2\pi}{(-2E)^{3/2}} < N\tau_\pm\right\}.$$

In particular, for energy levels $c < -3/2$, the bounded component of the

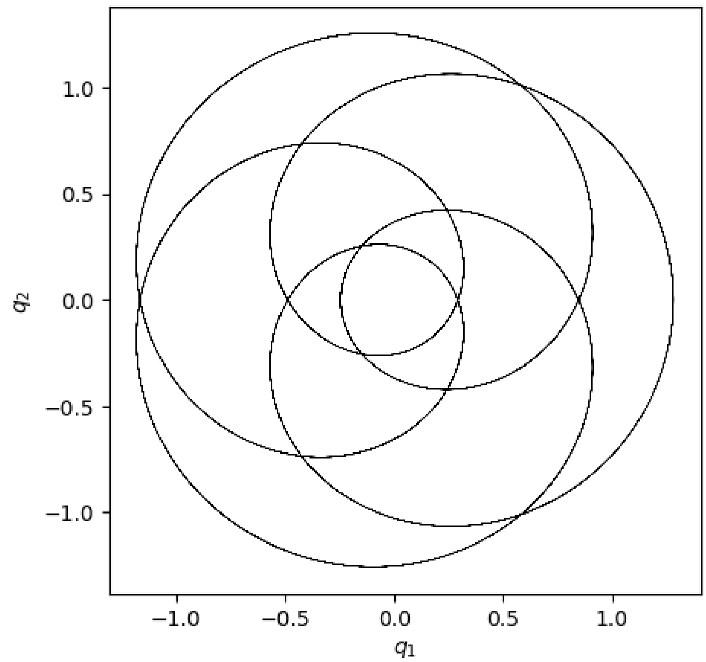

Figure 4.8: Planar periodic orbit with Kepler energy $E_{3,2}$ and initial conditions $q_2 = p_1 = 0$, $p_2 = 1/2$.

rotating Kepler problem is dynamically convex, and the doubly covered retrograde circular orbit is the only periodic orbit with a Conley-Zehnder index 3.

Remark 4.2.4. Since $c = E + L_3$, there exists a value of c between c^- and c^+ such that $L_3 = 0$. The corresponding orbits are the collision orbits.

4.3 Spatial Rotating Kepler Problem

In this section, we consider the Kepler problem adapted to the rotating frame around the q_3- and p_3-axes, which we will refer to as the **spatial rotating**

Kepler problem. The Hamiltonian is given by

$$\begin{aligned} H : T^*(\mathbb{R}^3 \setminus \{0\}) &\to \mathbb{R} \\ (q,p) &\mapsto \frac{1}{2}|p|^2 - \frac{1}{|q|} + (q_1p_2 - q_2p_1). \end{aligned}$$

The primary distinction here is the spatial dimension. We can also use the notion of an effective potential by introducing:

$$\begin{aligned} \tilde{p} &= (p_1 - q_2, p_2 + q_1, p_3) \\ U(q) &= -\frac{1}{|q|} - \frac{1}{2}|q|^2 \\ H(q,p) &= \frac{1}{2}|\tilde{p}|^2 + U(q). \end{aligned}$$

As before, we can write $H = E + L_3$ and $\{E, L_3\} = 0$, meaning that periodic orbit of H is a composition of a Kepler orbit and the L_3-flow.

4.3.1 Moser Regularization and Vertical Collision Orbits

We apply Moser regularization to the spatial rotating Kepler problem. As in Section 4.2, the Hamiltonian on T^*S^3 is given by

$$K_{\alpha,c}(x,y) = \frac{1}{2}|y|^2 \left(r + a(1 - x_0)(x_1y_2 - x_2y_1)\right)^2,$$

where $a = \alpha/r^2$ and $r = \sqrt{-2c}$.

Lemma 4.3.1. *The equations of motion for $K_{\alpha,c}$ are given by*

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{y}_0 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} f^2y_0 \\ f^2y_1 - a|y|^2fx_1x_2(1 - x_0) \\ f^2y_2 + a|y|^2fx_1x_2(1 - x_0) \\ f^2y_3 \\ |y|^2f(a(x_1y_2 - x_2y_1) - x_0(r + a(x_1y_2 - x_2y_1))) \\ |y|^2f(-ay_2(1 - x_0) - x_1(r + a(x_1y_2 - x_2y_1))) \\ |y|^2f(ay_1(1 - x_0) - x_2(r + a(x_1y_2 - x_2y_1))) \\ -|y|^2fx_3(r + a(x_1y_2 - x_2y_1)) \end{pmatrix}$$

where $f(x,y) = r + a(1-x_0)(x_1y_2 - x_2y_1)$

Proof. We can apply Lemma 4.1.9, where

$$\begin{aligned} f_x &= (-a(x_1y_2 - x_2y_1)), ay_2(1-x_0), -ay_1(1-x_0), 0) \\ f_y &= (0, -ax_2(1-x_0), ax_1(1-x_0), 0) \\ f_y \cdot x &= 0 \\ f + f_y \cdot y - f_x \cdot x &= r + a(x_1y_2 - x_2y_1). \end{aligned}$$

□

Solving this equation directly for a general solution is challenging. However, for a collision orbit with the initial condition

$$(p(0); q(0)) = (0, 0, P; 0, 0, Q),$$

the equation of motion simplify to

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_3 \\ \dot{y}_0 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} r^2 y_0 \\ r^2 y_3 \\ -x_0 \\ -x_3 \end{pmatrix}$$

with $x_1 = x_2 = y_1 = y_2 = 0$. These orbits are unaffected by angular momentum and correspond to the collision orbits of the (non-rotating) Kepler problem with initial conditions $q_3(0) = \mp 1/E$ and $p_3(0) = 0$, as described in Section 4.1.3. We denote these orbits by $\gamma_{c_\pm}$, and call **(positive/negative) vertical collision orbits**. These orbits are illustrated in Figure 4.9.

4.3.2 Moduli Space of Spatial Kepler Orbits

Recall that for fixed Kepler energy E, every Kepler orbit is an ellipse (including collision) of the same period. The regularized Kepler problem can be identified with the geodesic flow on T^*S^3. Hence the total orbit space must be ST^*S^3 which is diffeomorphic to $S^3 \times S^2$. Consequently, the moduli

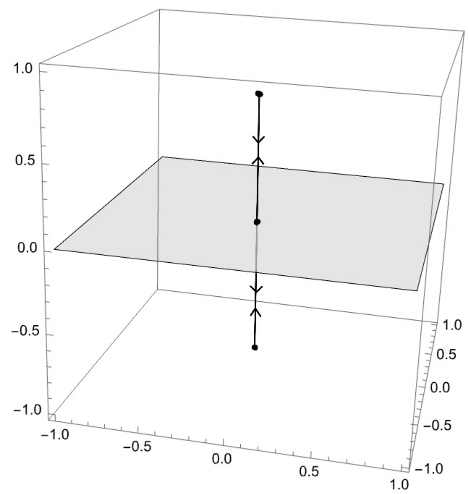

Figure 4.9: Illustration of vertical collision orbits.

space of orbits corresponds to the space of unit geodesics on S^3. A well-known result characterizes the space of geodesics on spheres.

Theorem 4.3.2. *The moduli space of unit geodesics on S^2 is diffeomorphic to S^2, and the moduli space of unit geodesics on S^3 is diffeomorphic to $S^2 \times S^2$.*

Proof. See [Bes78], Proposition 2.9. and 2.10. □

We'll rederive this result in a way adapted to the Kepler problem. We first consider the planar problem. Let $\mathcal{N}_E$ denote the moduli space of orbits for the planar Kepler problem with fixed E. Specifically, $\mathcal{N}_E$ is the level set of $K_E : T^*S^2 \to \mathbb{R}$, quotiented by S^1-action induced by the Hamiltonian flow. A planar orbit is characterized by:

1. The rotation direction (determined by sign of L_3),
2. The major axis (determined by the direction of (A_1, A_2)),
3. The eccentricity (given by $\varepsilon^2 = |A|^2 = 2EL_3^2 + 1$).

Using the formula for the eccentricity, we have

$$A_1^2 + A_2^2 + (\sqrt{-2E}L_3)^2 = 1.$$

This implies
$$\mathcal{N}_E = \left\{(A_1, A_2, \sqrt{-2E}L_3)\right\} \simeq S^2.$$
The retrograde and direct orbits correspond to the points
$$\gamma_\pm = (0, 0, \pm 1).$$

The total energy $c = E + L_3$ is determined by L_3-component, meaing that c serves as a *height function* on $\mathcal{N}_E$, which is a Morse function. The regular level sets of c is S^1-families of ellipses that are invariant under the L_3-action. The retrograde and direct orbits correspond to the maximum and minimum of the total energy, respectively.

Now let $\mathcal{M}_E$ denote the moduli space of orbits for the regularized spatial Kepler problem with Kepler energy E. Specifically, $\mathcal{M}_E$ is the level set of $K_E : T^*S^3 \to \mathbb{R}$, quotiented by S^1-action induced by the Hamiltonian flow.

To determine an ellipse in $\mathbb{R}^3$ with one focus at the origin, we need to specify:

1. The plane containing the ellipse and rotation direction (given by L),
2. The major axis (given by A),
3. The eccentricity (given by $\varepsilon^2 = |A|^2 = 2EL^2 + 1$).

Thus, the angular momentum L and the Laplace-Runge-Lenz vector A, together with E, uniquely characterizes the Kepler orbit.

Lemma 4.3.3. *A map* Φ *defined as*
$$\begin{aligned}\Phi : \mathcal{M}_E &\to S^2 \times S^2 \\ \gamma &\mapsto \left(\sqrt{-2E}L - A, \sqrt{-2E}L + A\right)\end{aligned}$$
is a well-defined bijection.

Proof. Let $(x, y) = \left(\sqrt{-2E}L - A, \sqrt{-2E}L + A\right)$. We verify
$$|x|^2 = -2EL^2 + A^2 - \sqrt{-2E}L \cdot A = -2EL^2 + A^2 = 1,$$
$$|y|^2 = -2EL^2 + A^2 + \sqrt{-2E}L \cdot A = -2EL^2 + A^2 = 1.$$

Hence, $(x, y) \in S^2 \times S^2$. Given (x, y), we reconstruct L and A by

$$L = \frac{x+y}{2\sqrt{-2E}}, \qquad -A = \frac{x-y}{2}.$$

Using L and A, the orbit γ can be reconstructed as decribed in Theorem 4.1.4. □

This result matches Theorem 4.3.2, providing explicit coordinates (x, y) on $\mathcal{M}_E$. We note some properties of this parametrization.

1. **Eccentricity**: $\varepsilon = |A| = |x - y|/2$.
2. **Circular orbits**: Circular orbits correspond to the diagonal $\{x = y\}$, which is $A = 0$, forming an S^2-family.
3. **Planar Orbits**: Planar orbits satisfy $L_1 = L_2 = A_3 = 0$, giving

$$\mathcal{N}_E = \left\{(x, y) \in S^2 \times S^2 \,:\, x_1 = -y_1,\, x_2 = -y_2,\, x_3 = y_3\right\} \simeq S^2.$$

4. **Retrograde and direct orbits**: These correspond to

$$\begin{aligned}\gamma_+ &= ((0,0,1),(0,0,1))\\ \gamma_- &= ((0,0,-1),(0,0,-1)).\end{aligned}$$

5. **Collision Orbits**: Collision orbits correspond to the anti-diagonal $\{x = -y\}$, which is $L = 0$, also forming an S^2-family.
6. **Vertical collision orbits**: These correspond to

$$\begin{aligned}\gamma_{c_+} &= ((0,0,1),(0,0,-1))\\ \gamma_{c_-} &= ((0,0,-1),(0,0,1))\end{aligned}$$

To discuss the rotating Kepler problem with this framework, we need to consider the total energy $c = E + L_3$. First, we analyze the S^1-action corresponding to L_3, which represents rotations along q_3- and p_3-axis. The action on $\mathcal{M}_E$ is given by

$$\theta \cdot (x, y) = \Big(e^{i\theta}(x_1 + ix_2), x_3, e^{i\theta}(y_1 + iy_2), y_3\Big)$$

where the q_1q_2-plane and p_1p_2-plane are identified with $\mathbb{C}$ for convenience. This action is free except at four points: $\gamma_\pm$ and $\gamma_{c\pm}$.

For a fixed Kepler energy E, varying c is equivalent to varying L_3. Using the coordinates (x, y) of $\mathcal{M}_E$, L_3 is determined by $x_3 + y_3$. Denoting this function as f, we define

$$\begin{aligned} f : S^2 \times S^2 &\to [-2, 2] \\ (x, y) &\mapsto x_3 + y_3. \end{aligned}$$

There are exactly four critical points of f, corresponding to $\gamma_\pm$ and $\gamma_{c\pm}$. These are precisely the fixed points of the S^1-action associated with L_3, a consequence of Noether's theorem.

The function f is a Morse function; γ_- is a minimum with index 0, γ_+ is a maximum with index 4, and $\gamma_{c\pm}$ are saddle points with index 2. From the perspective of Morse homology or handle attachment, the level sets of f are described as follows:

$$f^{-1}(a) = \begin{cases} \{\gamma_-\} & \text{if } a = -2 \\ S^3 & \text{if } -2 < a < 0 \\ S & \text{if } a = 0 \\ S^3 & \text{if } 0 < a < 2 \\ \{\gamma_+\} & \text{if } a = 2 \end{cases}$$

where S is homeomorphic to ΣT^2, the suspension of T^2. To better understand this, define the function

$$\begin{aligned} g : S &\to [-2, 2] \\ (x, y) &\mapsto x_3 - y_3 \end{aligned}$$

which is proportional to A_3. The level sets of g are given by

$$g^{-1}(b) = \begin{cases} \{\gamma_{c_-}\} & \text{if } b = -2 \\ T^2 & \text{if } -2 < b < 2 \\ \{\gamma_{c_+}\} & \text{if } b = 2 \end{cases}$$

Note 4.3.4. The space ΣT^2 is homotopy equivalent to $S^3 \vee S^2 \vee S^2$. In particular, it is not a manifold, as the link of the vertices is T^2, not a sphere.

Note that for $0 \neq a \in (-1, 1)$, we can similarly define

$$\begin{aligned} g : f^{-1}(a) &\to [-(2 - |a|), 2 - |a|] \\ (x, y) &\mapsto x_3 - y_3. \end{aligned}$$

where the level sets are

$$g^{-1}(b) = \begin{cases} S^1 & \text{if } b = -(2 - |a|) \\ T^2 & \text{if } -(2 - |a|) < b < 2 - |a| \\ S^1 & \text{if } b = 2 - |a| \end{cases}$$

This provides a Heegaard decomposition of S^3 for each energy level set S^3.

These observations lead to the definition of the following map

$$\begin{aligned} F : \mathcal{M}_E &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x_3, y_3) \end{aligned}$$

The map F has the following properties:

1. F has exactly 4 critical points, $\gamma_\pm$ and $\gamma_{c_\pm}$.

2. Edges of the picture correspond to the spaces

$$\begin{aligned} \overline{\gamma_- \gamma_{c_+}} &= (0, 0, -1) \times S^2, \\ \overline{\gamma_- \gamma_{c_-}} &= S^2 \times (0, 0, -1) \\ \overline{\gamma_{c_+} \gamma_+} &= S^2 \times (0, 0, 1) \\ \overline{\gamma_{c_-} \gamma_+} &= (0, 0, 1) \times S^2 \end{aligned}$$

3. The edges are the singular locus of F, where dF has rank 1, except for 4 endpoints, where dF has rank 0.

4. The fiber of F for interior points on the edges is S^1.

5. All other points are regular, with fibers $S^1 \times S^1$.

This is the standard toric diagram of $S^2 \times S^2$, illustrated in Figure 4.10

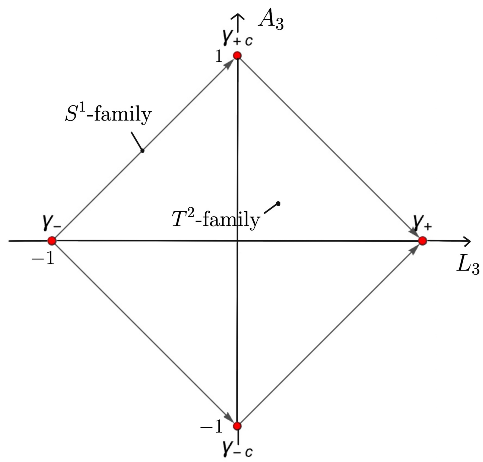

Figure 4.10: The toric picture of $\mathcal{M}_E$.

4.3.3 Periodic Orbits

The period of a Kepler orbit with Kepler energy E is given by

$$\tau = \frac{2\pi}{(-2E)^{3/2}}.$$

As in the planar case, the orbit of the spatial rotating Kepler problem is the composition of $Fl_t^{X_E}$ and $Fl_t^{X_{L_3}}$.

There are two planar circular orbits, the retrograde orbit γ_+ and the direct orbit γ_-, which are periodic for any Kepler energy. Similarly, the vertical collision orbits $\gamma_{c\pm}$ are not effected by L_3-action, making them periodic for any Kepler energy as well.

In general, as in the planar case, the Kepler orbit must have period $2\pi(k/l)$ to be periodic for the rotating Kepler problem. This requires that the Kepler energy satisfies

$$E = E_{k,l} = -\frac{1}{2}\left(\frac{k}{l}\right)^{2/3}.$$

for some $k, l \in \mathbb{N}$. An example is illustrated in Figure 4.11. This implies that at the energy level $E_{k,l}$, periodic orbits emerge as a family parameterized by $\mathcal{M}_E$. The situation is depicted in Figure 4.12[2].

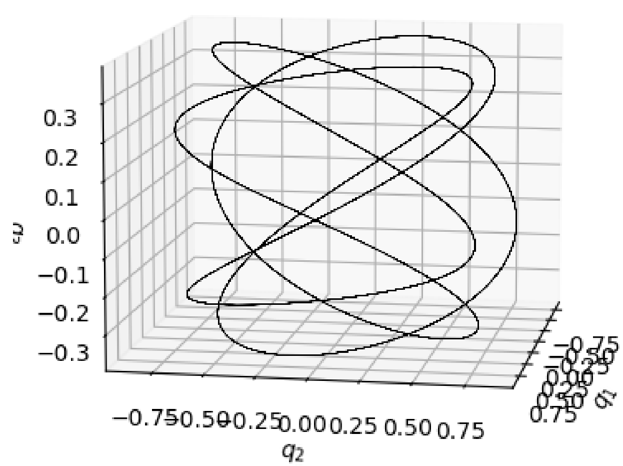

Figure 4.11: A spatial periodic orbit of Kepler energy $E_{3,2}$, initial conditions $q_2 = q_3 = p_1 = 0$ and $(p_2, p_2) = (\sqrt{3}/2, 1/2)$.

If we fix the total energy level $c < -3/2$, the periodic orbits contained in the level set $H^{-1}(c)$ are as follows. This scenario is illustrated in Figure 4.13.

[2]We thank Chankyu Joung for providing the illustration.

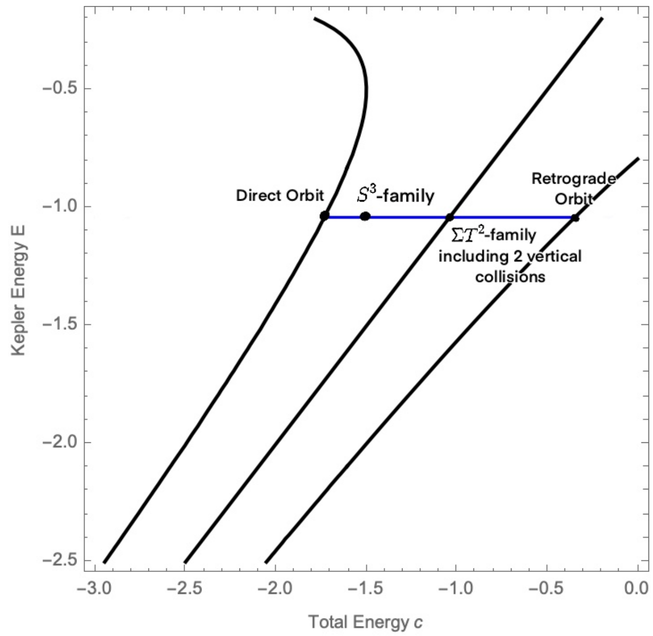

Figure 4.12: The family of orbits at Kepler energy $E_{3,2}$.

1. The retrograde orbit γ_+ : This has Kepler energy E_+ satisfying

$$E_+ + \frac{1}{\sqrt{-2E_+}} = c.$$

2. The direct orbit γ_- : This has Kepler energy E_- satisfying

$$E_- - \frac{1}{\sqrt{-2E_-}} = c.$$

3. Two vertical collision orbits $\gamma_{c_\pm}$: These have Kepler energy $E = c$.

4. The S^3-family of elliptic orbits : These orbits correspond to Kepler energies $E_{k,l}$ for all k, l such that

$$E_- < E_{k,l} < E_+.$$

5. The ΣT^2-family of orbits : The family appears only if $c = E_{k,l}$ for

some k, l. This family includes two vertical collision orbits.

6. Multiple covers of above orbits.

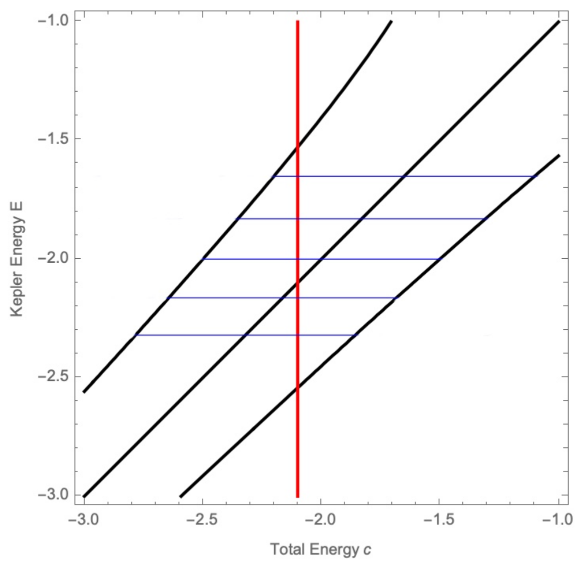

Figure 4.13: A family of orbits with total energy $c = -2.1$. The figure includes the direct orbit, two vertical collisions, and the retrograde orbit. The blue lines correspond to $E_{6,1}, E_{7,1}, E_{8,1}, E_{9,1}, E_{10,1}$. For each intersection, there exists an S^3-family of orbits. Note that there are infinitely many $E_{k,l}$ values not shown in the diagram.

4.4 Conley-Zehnder Index of Periodic Orbits

In this section, we compute the Conley-Zehnder index of periodic orbits in the spatial Kepler problem. We begin by determining the indices of the multiple covers of the non-degenerate retrograde, direct, and vertical collision orbits and then consider the orbits with Kepler energy $E_{k,l}$.

4.4.1 Index of Retrograde and Direct Orbits

Following the strategy of [AFFvK13], as described in Proposition 4.2.2, we use cylindrical coordinates to treat the planar circular orbits $\gamma_\pm$. Consider the coordinate transformation

$$(q_1, q_2, q_3) = (r\cos\theta, r\sin\theta, z)$$
$$(p_1, p_2, p_3) = (p_r\cos\theta - \frac{p_\theta}{r}\sin\theta, p_r\sin\theta + \frac{p_\theta}{r}\cos\theta, p_z)$$

where p_r, p_θ, p_z are defined such that

$$p_1 dq_1 + p_2 dq_2 + p_3 dq_3 = p_r dq_r + p_\theta d\theta + p_z dz.$$

Note that $q_1 p_2 - q_2 p_1 = p_\theta$. In these coordinates, the Hamiltonian for the rotating Kepler problem is given by

$$H(r, \theta, z, p_r, p_\theta, p_z) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + p_\theta - \frac{1}{\sqrt{r^2 + z^2}}.$$

and the corresponding Hamiltonian vector field is

$$X_H = p_r\partial_r + \left(\frac{p_\theta}{r^2} + 1\right)\partial_\theta + p_z\partial_z + \left(\frac{p_\theta^2}{r^3} - \frac{r}{(r^2+z^2)^{3/2}}\right)\partial_{p_r} - \frac{z}{(r^2+z^2)^{3/2}}\partial_{p_z}$$

We impose the conditions for the planar circular orbits.

1. **Planar** : $z = p_z = 0$.
2. **Circular** : $r = r_0$ is constant. From this, it follows that $\dot{r} = p_r = 0$ and $\dot{p}_r = 0$. Combined with $z = 0$, this implies $p_\theta^2 = r_0$. We take $p_\theta = \pm\sqrt{r_0}$. For shorthand, we write $p_\theta = \omega_0$.

Under these conditions, the Hamiltonian vector field simplifies to

$$X_H = \left(\frac{1}{\omega_0^3 + 1}\right)\partial_\theta$$

and the orbit can be expressed as

$$\begin{pmatrix} r(t) \\ \theta(t) \\ z(t) \\ p_r(t) \\ p_\theta(t) \\ p_z(t) \end{pmatrix} = \begin{pmatrix} \omega_0^2 \\ \left(\frac{1}{\omega_0^3}+1\right)t \\ 0 \\ 0 \\ \omega_0 \\ 0 \end{pmatrix}.$$

As stated in Proposition 4.2.2, $\omega_0 = \pm\sqrt{r_0}$ correspond to $\gamma_\pm$ respectively, with a period given by

$$\tau_\pm = \pm\frac{2\pi}{1/\omega_0^3+1}.$$

Since $r_0 = -1/2E$ holds here as well, we can write

$$\tau_\pm = \frac{2\pi}{(-2E)^{3/2}\pm 1}$$

assuming that $E < -1/2$.

From Proposition 2.5.4, the Conley-Zehnder index can be computed using the linearized Hamiltonian flow. Denote the perturbation vector by $\Delta = (\Delta r, \Delta\theta, \Delta z, \Delta p_r, \Delta p_\theta, \Delta p_z)$. The linearized flow is governed by

$$\dot{\Delta} = \mathbf{L}\Delta$$

where $\mathbf{L}$ is the linearization matrix obtained by differentiating X_H,

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -2p_\theta/r^3 & 0 & 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -3p_\theta^2/r^4 + 3r^2/R^5 - 1/R^3 & 0 & 3rz/R^5 & 0 & 2p_\theta/r^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3rz/R^5 & 0 & 3z^2/R^5 - 1/R^3 & 0 & 0 & 0 \end{pmatrix}.$$

Here, $R = \sqrt{r^2 + z^2}$. After substituting the orbit $\gamma_\pm$, we find

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -2/\omega_0^5 & 0 & 0 & 0 & 1/\omega_0^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/\omega_0^6 & 0 & 0 & 0 & 2/\omega_0^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\omega_0^6 & 0 & 0 & 0 \end{pmatrix}.$$

We need to determine the frame of $\ker(dH) \cap \ker(\lambda)$ along the orbit. We have

$$\begin{aligned} dH = &\left(-\frac{p_\theta^2}{r^3} + \frac{r}{(r^2+z^2)^{3/2}}\right) dr + \frac{z}{(r^2+z^2)^{3/2}} dz \\ &+ p_r dp_r + \left(1 + \frac{p_\theta}{r^2}\right) dp_\theta + p_z dp_z \end{aligned}$$

which simplifies to

$$dH = \left(1 + \frac{1}{\omega_0^3}\right) dp_\theta$$

along the orbit. The contact form is given by

$$\lambda = -qdp = p_\theta d\theta - r dp_r - z dp_z$$

which simplifies to

$$\lambda = \omega_0 d\theta - \omega_0^2 dp_r$$

along the orbit.

Note 4.4.1. Here, we use $-qdp$ instead of the standard pdq, as in the context of Moser regularization, the roles of p and q are interchanged. Specifically, let $X = q\partial_q$ so that $i_X\omega = -qdp$. Note that $X(H) = 1/|q| + L_3$, and $X(H)$ must be positive. This can be seen as a special case of a similar phenomenon in the restricted three-body problem. See Theorem 5.2.1 in [CJK20].

With this, we can find a symplectic frame

$$(X_1, X_2, X_3, X_4) = \left(\partial_\theta + \frac{1}{\omega_0}\partial_{p_r}, \omega_0\partial_r, \partial_{p_z}, \partial_z\right)$$

where $\omega(X_1, X_2) = \omega(X_3, X_4) = 1$ and $\omega(X_i, X_j) = 0$.

Lemma 4.4.2 ([AFFvK13], Appendix B)**.** *Let $T^*S^2 \subset T^*\mathbb{R}^3$ with coordinates (x, y) such that $|x|^2 = 1$ and $\langle x, y\rangle = 0$. Let $K : T^*S^2 \to \mathbb{R}$ be a fiberwise star-shaped Hamiltonian such that $y\partial_y K > 0$, and let the contact form be given by $\lambda = ydx$. Then,*

$$X_1 = \left(y \times x - \frac{(y \times x) \cdot \partial_y K}{y \cdot \partial_y K} \right) \cdot \partial_y$$
$$X_2 = -\frac{(y \times x) \cdot \partial_x K}{y \cdot \partial_y K} y \cdot \partial_y + (y \times x) \cdot \partial_x$$

provides a global trivialization of $\ker \lambda \cap \ker dK$.

Lemma 4.4.3. *The framing X_1, X_2, X_3, X_4 given above can be extended to a capping disk.*

Proof. From [AFFvK13] Appendix B, X_1, X_2 corresponds to the framing in the planar orbit given in the previous lemma. Thus, X_1, X_2 can be extended to a planar capping disk. Since X_3, X_4 do not involve planar coordinates, they can also be extended to the capping disk. □

We also take a normal frame

$$(N_1, N_2) = \left(\frac{1}{\omega_0} \partial_\theta, \omega_0 \partial_{p_\theta} + \omega_0^2 \partial_r \right)$$

such that $\omega(N_1, N_2) = 1$ and $\omega(X_i, N_j) = 0$. Note that N_1 represents the Reeb direction of λ. We have

$$\begin{aligned}
\mathbf{L}X_1 &= (1/\omega_0)\partial_r = (1/\omega_0^2)X_2, \\
\mathbf{L}X_2 &= -(2/\omega_0^4)\partial_\theta - (1/\omega_0^5)\partial_{p_r} = -(1/\omega_0^4)X_1 - (1/\omega_0^3)N_1, \\
\mathbf{L}X_3 &= \partial_z = X_4, \\
\mathbf{L}X_4 &= -(1/\omega_0^6)\partial_{p_z} = -(1/\omega_0^6)X_3.
\end{aligned}$$

The linearized flow matrix under the frame (X_1, X_2, X_3, X_4) is

$$\mathbf{L} = \begin{pmatrix} 0 & -1/\omega_0^4 & 0 & 0 \\ 1/\omega_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\omega_0^6 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

By taking the matrix exponential, we obtain the path of symplectic matrices

$$\Psi_H(t) = \begin{pmatrix} \cos(t/\omega_0^3) & -(1/\omega_0)\sin(t/\omega_0^3) & 0 & 0 \\ \omega_0 \sin(t/\omega_0^3) & \cos(t/\omega_0^3) & 0 & 0 \\ 0 & 0 & \cos(t/\omega_0^3) & -(1/\omega_0^3)\sin(t/\omega_0^3) \\ 0 & 0 & \omega_0^3 \sin(t/\omega_0^3) & \cos(t/\omega_0^3) \end{pmatrix}$$

The crossings occur exactly at $2\pi\omega_0^3\mathbb{Z}$. The crossing form is given by $\Omega\dot{\Psi}(t)$ where Ω is defined in Section 2.5.1,

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We have

$$\Omega\dot{\Psi}_H(t) = \Omega\mathbf{L} = \begin{pmatrix} 1/\omega_0^2 & 0 & 0 & 0 \\ 0 & 1/\omega_0^4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\omega_0^6 \end{pmatrix}$$

so the signature of the crossing forms is always 4. Using $\omega_0 = \pm 1/\sqrt{-2E}$, we derive the following result.

Proposition 4.4.4. *Let $\gamma_\pm$ denote the retrograde and direct orbits of Kepler energy E, where $E \neq E_{k,l}$ for any k, l. Then, $\gamma_\pm$ and their multiple covers* *are non-degenerate. The Conley-Zehnder index of N-th iterate of $\gamma_\pm$ is*

$$\begin{aligned}\mu_{CZ}(\gamma_\pm^N) &= 2 + 4\max\left\{n \in \mathbb{Z}_{>0} : \frac{2\pi n}{(-2E)^{3/2}} < N\frac{2\pi}{(-2E)^{3/2} \pm 1}.\right\} \\ &= 2 + 4\max\left\{n \in \mathbb{Z}_{>0} : n < N\frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}\right\} \\ &= 2 + 4\left\lfloor N\frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}\right\rfloor\end{aligned}$$

Note 4.4.5. This is exactly twice the index of the retrograde and direct orbits in the planar problem, which was computed in [AFFvK13] and introduced in Theorem 4.2.3.

We write

$$\mu_\pm = \mu_\pm(E) = \frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}.$$

The functions $\mu_\pm$ are illustrated in Figure 4.14. Here's some observation about the indices of $\gamma_\pm$.

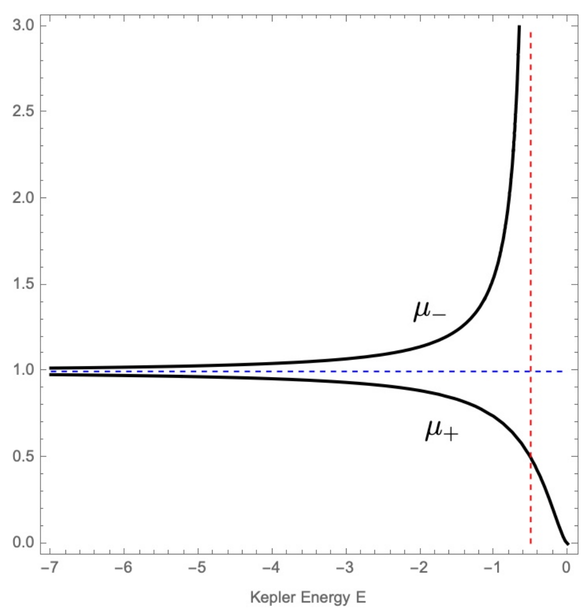

Figure 4.14: The graph of $\mu_\pm$.

1. **Behavior for small** E : For given N, there exists a sufficiently small E such that $N\mu_+ < 1$ and $1 < N\mu_- < 2$. Hence,

$$\mu_{CZ}(\gamma_+^N) = 4N - 2$$
$$\mu_{CZ}(\gamma_-^N) = 4N + 2$$

for every small E.

2. **Decreasing behavior of retrograde orbits** : The index $\mu_{CZ}(\gamma_+^N)$ decreases by 4 each time μ_+ touches k/N for some $k = 1, \cdots, N-1$. The corresponding energies satisfy

$$\frac{(-2E)^{3/2}}{(-2E)^{3/2}+1} = \frac{k}{N}$$

or equivalently

$$E = E_{k,N-k} = -\frac{1}{2}\left(\frac{k}{N-k}\right)^{2/3}$$

3. **Increasing behavior of direct orbits** : The index $\mu_{CZ}(\mu_-^N)$ increases by 4 each time μ_- touches $1 + k/N$ for some $k = 1, 2, \cdots$. The corresponding energies satisfy

$$E = E_{N+k,k} = -\frac{1}{2}\left(\frac{N+k}{k}\right)^{2/3}.$$

We summarize the result as follows.

Theorem 4.4.6. *For fixed N, the Conley-Zehnder index of N-th cover of retrograde orbit with Kepler energy E is:*

$$\mu_{CZ}(\gamma_+^N) = \begin{cases} 4N-2 & \text{if } E < E_{N-1,1} \\ 4(N-k)-2 & \text{if } E_{N-k,k} < E < E_{N-k-1,k+1} \\ & \text{for } k = 1, 2, \cdots, N-2 \\ 2 & \text{if } E > E_{1,N-1} \end{cases}$$

In particular, the simple retrograde orbit has always index 2.

Similarly, the Conley-Zehnder index of N-th cover of the direct orbit is:

$$\mu_{CZ}(\gamma_-^N) = \begin{cases} 4N+2 & \text{if } E < E_{N+1,1} \\ 4(N+k)+2 & \text{if } E_{N+k,k} < E < E_{N+k+1,k+1} \\ & \text{for } k = 1, 2, \cdots, \end{cases}$$

In particular, the indices of direct orbits diverge as $E \to -1/2$.

4.4.2 Index of Vertical Collision Orbits

For the vertical collision orbits, we use the result of Lemma 4.3.1. We impose the condition of vertical collision orbits, which is given by

$$\begin{aligned}\gamma_{c\pm}(t) &= (x(t), y(t)) \\ &= (-\cos(rt), 0, 0, \pm\sin(rt), (1/r)\sin(rt), 0, 0, \pm(1/r)\cos(rt))\end{aligned}$$

in T^*S^3. In this case, we have $x_1 = x_2 = y_1 = y_2 = 0$, so that $f(x, y) = r$ and $|y| = 1/r$ in Lemma 4.3.1.

Lemma 4.4.7. *Along the vertical collision orbit in T^*S^3, we can take*

$$(X_1, X_2, X_3, X_4) = (\partial_{y_1}, \partial_{x_1}, \partial_{y_2}, \partial_{x_2})$$

as a symplectic frame of the contact structure of $K^{-1}(1/2)$.

Proof. We can take $y \cdot \partial_y$ as a Liouville vector field, so $\lambda = i_X dy \wedge dx = ydx$ as a contact form. It is clear that X_i are tangent to T^*S^3 along the vertical collision orbit. Additionally,

$$\begin{aligned} dK &= r^2 y_0 dy_0 + r^2 y_3 dy_3, \\ \lambda &= y_0 dx_0 + y_3 dx_3 \end{aligned}$$

along the orbit, so $X_i \in \ker\lambda \cap \ker dK$. Finally, $\omega(X_1, X_2) = \omega(X_3, X_4) = 1$ and $\omega(X_i, X_j) = 0$ otherwise, verifying that (X_1, X_2, X_3, X_4) forms a symplectic frame. □

Lemma 4.4.8. *The symplectic frame given in Lemma 4.4.7 is valid along γ_c*

regarded as an orbit of K_E. The frame can also be extended to a symplectic frame on a capping disk of γ_c in $K_E^{-1}(1/2)$.

Proof. For simplicity, assume $r = 1$. Consider the cotangent bundle of the subspace

$$S = \{(\cos t\cos\theta, \sin\theta, 0, \sin t\cos\theta)\} \subset S^3.$$

Here, T^*S is a subset of $T^*\mathbb{R}^3 = \{x_2 = y_2 = 0\}$. Applying Lemma 4.4.2 to the (x_0, x_1, x_3)-coordinate and the Hamiltonian $K = |y|^2/2$, we observe that X_1, X_2 match the frame given in the lemma along the orbit. If we take a capping disk D in $ST^*S = K^{-1}(1/2) \cap T^*S$, as in Lemma 4.4.3, X_3, X_4 can be extended to D. □

Now, we calculate the linearized flow by differentiating Hamiltonian equation

$$\begin{pmatrix} \dot{y}_1 \\ \dot{x}_1 \\ \dot{y}_2 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} |y|^2 f(-ay_2(1-x_0) - x_1(r + a(x_1y_2 - x_2y_1))) \\ f^2y_1 - a|y|^2 f x_1x_2(1-x_0) \\ |y|^2 f(ay_1(1-x_0) - x_2(r + a(x_1y_2 - x_2y_1))) \\ f^2y_2 + a|y|^2 f x_1x_2(1-x_0) \end{pmatrix}.$$

After substituting $x_1 = x_2 = y_1 = y_2 = 0$, we find

$$\mathbf{L} = \begin{pmatrix} 0 & -r|y|^2 f & -a|y|^2 f(1-x_0) & 0 \\ f^2 & 0 & 0 & 0 \\ a|y|^2 f(1-x_0) & 0 & 0 & -r|y|^2 f \\ 0 & 0 & f^2 & 0 \end{pmatrix}.$$

Using $f = r$, $|y| = 1/r$ and $1 - x_0 = 1 + \cos(rt)$, we get

$$\mathbf{L} = \begin{pmatrix} 0 & -1 & -a(1+\cos(rt))/r & 0 \\ r^2 & 0 & 0 & 0 \\ a(1+\cos(rt))/r & 0 & 0 & -1 \\ 0 & 0 & r^2 & 0 \end{pmatrix}$$

Since $\mathbf{L}$ is time-dependent, directly integrating it is challenging. Instead, we use the following lemma to compute the Conley-Zehnder index.

Lemma 4.4.9. *Let K_E be the regularized Hamiltonian of the (non-rotating) Kepler problem given in Section 4.1.3. Let Ψ_{K_E} and Ψ_{L_3} be paths of symplectic matrices given by the linearized flow of K_E and L_3. Then,*

$$\mu_{CZ}(\gamma_c) = \mu_{RS}(\Psi_{K_E}) + \mu_{RS}(\Psi_{L_3}).$$

Proof. See the end of this subsection. □

By setting $a = 0$ in $\mathbf{L}$ we find the linearized flow of K_E. After integration, the resulting path of symplectic matrices is

$$\Psi_{K_E}(t) = \begin{pmatrix} \cos(rt) & \sin(rt)/r & 0 & 0 \\ -r\sin(rt) & \cos(rt) & 0 & 0 \\ 0 & 0 & \cos(rt) & \sin(rt)/r \\ 0 & 0 & -r\sin(rt) & \cos(rt) \end{pmatrix}$$

Notice that the period of Ψ_{K_E} is equal to the period of the collision orbit. The crossing occurs at $t = (2\pi/r)\mathbb{Z}$, and the crossing form is

$$\Omega\dot{\Psi}_{K_E}(\tau) = \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that the signature of the crossing form is always 4.

Next, we compute the index of the orbit of the angular momentum L_3 on $T^*\mathbb{R}^3$. The Hamiltonian equation is given by

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{pmatrix} = \begin{pmatrix} -q_2 \\ q_1 \\ 0 \\ -p_2 \\ p_1 \\ 0 \end{pmatrix}$$

and the linearization matrix with respect to the symplectic basis $\partial_{p_1}, \partial_{q_1}, \partial_{p_2}, \partial_{q_2}$

is

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The linearized flow is given by

$$\Psi_L(t) = \begin{pmatrix} \cos t & 0 & \sin t & 0 \\ 0 & \cos t & 0 & \sin t \\ -\sin t & 0 & \cos t & 0 \\ 0 & -\sin t & 0 & \cos t \end{pmatrix}.$$

The crossing occurs at $2\pi\mathbb{Z}$, and the crossing form is

$$\Omega\dot{\Psi}_L(\tau) = \Omega\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

so the signature of the crossing form is zero. Combining these results, we have the following.

Proposition 4.4.10. *Let $\gamma_{c\pm}$ be the vertical collision orbits of Kepler energy E, where $E \neq E_{k,l}$ for any k, l. Then $\gamma_{c\pm}$ and their multiple covers are non-degenerate. The Conley-Zehnder index of the N-th iterate of $\gamma_{c\pm}$ is*

$$\begin{aligned}\mu_{CZ}(\gamma_{c\pm}^N) &= \mu_{RS}(\Psi_{K_E}) + \mu_{RS}(\Psi_L) \\ &= (2 + 4(N-1) + 2) + 0 = 4N\end{aligned}$$

In particular, the index of the multiple cover of vertical collision orbits never changes.

Proof of Lemma 4.4.9

Let $H = E + L_3$ be the Hamiltonian of rotating Kepler problem, and let K be the regularized Hamiltonian. We denote the regularized (non-rotating)

Kepler Hamiltonian by K_E. Let $\gamma : [0, \tau] \to T^*S^3$ be a K-orbit. Since the flow of K and H are parallel, with Lemma 2.5.3 we have

$$\mu_{CZ}(\gamma) = \mu_{RS}(\Psi_K) = \mu_{RS}(\Psi_H)$$

except for the collision orbits, which H is not defined.

We first extend Ψ_H to the collision locus. Let $\gamma : [0, \tau] \to T^*S^3$ be a K-orbit such that $\gamma(0)$ lies on the collision locus. Let $A : \gamma^*\xi \to [0, \tau] \times \mathbb{R}^4$ be a trivialization of the contact structure along γ. For $\varepsilon > 0$, find an H-orbit $\tilde{\gamma}_\varepsilon : [0, \sigma(\varepsilon)] \to T^*\mathbb{R}^3$, which is a reparametrization of $\gamma|_{[\varepsilon,\tau]}$. Under the inverse stereographic projection, we can regard $\tilde{\gamma}_\varepsilon$ as an orbit on T^*S^3. We define

$$\Phi_\tau^\varepsilon = A(t)^{-1} dFl_{\sigma(\varepsilon)}^{X_H}|_\xi A(\varepsilon) \in Sp(4),$$
$$\Psi_\varepsilon = A(\varepsilon)^{-1} dFl_\varepsilon^{X_K}|_\xi A(0) \in Sp(4)$$

Then we have $\Psi_\tau = \Phi_\tau^\varepsilon \Psi_\varepsilon$. It's clear that $\lim_{\varepsilon\to 0} \Psi_\varepsilon = \Psi_0 = \mathrm{Id}$, so that

$$\lim_{\varepsilon\to 0} \Phi_\tau^\varepsilon = \Psi_\tau.$$

In a similar way, we can extend Ψ_E to the collision locus.

Now since $H = E + L$ and $\{E, L\} = 0$, we have

$$dFl_t^{X_H} = dFl_t^{X_{L_3}} \circ dFl_t^{X_E}$$

If we restrict ourselves to the collision orbit, we get

$$A(t) dFl_t^{X_H}|_\xi A(0)^{-1} = \left(A(t) dFl_t^{X_{L_3}}|_\xi A(t)^{-1}\right) \left(A(t) dFl_t^{X_E}|_\xi A(0)^{-1}\right).$$

From Theorem 2.5.1, we know that

$$\mu_{RS}(\Psi_H) = \mu_{RS}(\Psi_{L_3}) + \mu_{RS}(\Psi_E).$$

Since E-orbit is a reparamerization of K_E-orbit, we have $\mu_{RS}(\Psi_E) = \mu_{RS}(\Psi_{K_E})$, which leads us to the conclusion. Moreover, since the L_3-orbit with initial conditions on the collision orbit is constant, we can compute the index with respect to any frame.

4.4.3 Relation with Symplectic Homology

With the rotating Kepler Hamiltonian $H : T^*S^3 \to \mathbb{R}$, we consider the symplectic homology of T^*S^3. Viterbo's theorem (Theorem 2.5.6) states that the symplectic homology of T^*S^3 is isomorphic to the loop space homology of S^3.

Proposition 4.4.11. *Let ΛS^3 denote the free loop space of S^3. Then,*

$$H_*(\Lambda S^3, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } * = 0 \text{ or } * \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is well-known, for example, from [BT82], that the homology of the based loop space ΩS^3 of S^3 is given by

$$H_*(\Omega S^3, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } * = 2k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have a fibration $\Omega S^3 \to \Lambda S^3 \to S^3$, which is trivial since S^3 is a Lie group. Applying the Künneth formula yields the result. □

By Viterbo's theorem, $SH_*(ST^*S^3) \simeq H^*(\Lambda S^3, \mathbb{Z}_2)$. Additionally, we have the following result regarding S^1-equivariant symplectic cohomology.

Proposition 4.4.12 ([KvK16], Proposition 5.12)**.** *The $+$-part of the S^1-equivariant symplectic homology of T^*S^3 is given by*

$$SH_*^{S^1,+}(T^*S^3) = \begin{cases} \mathbb{Z}_2 & \text{if } * = 2 \\ \mathbb{Z}_2^2 & \text{if } * = 2\text{k} \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

We now relate our result to Proposition 4.4.12. Fix $N \in \mathbb{N}$. Let $c < -3/2$ with $c \neq E_{k,l}$ for any k, l. Denote

$$c_{k,l}^{\pm} = E_{k,l} \pm \frac{1}{\sqrt{-2E_{k,l}}}$$

for the total energy of the retrograde orbit and direct orbit with Kepler energy $E_{k,l}$. There exists an energy level $c \ll -3/2$ sufficiently small such that:

1. $\mu_{CZ}(\gamma_+^k) = 4k - 2$ for $k \leq N$.
2. $\mu_{CZ}(\gamma_-^k) = 4k + 2$ for $k \leq N$.
3. Higher iterates of the retrograde orbit have index$\geq 4N - 2$.

This condition is achieved by taking c smaller thann $c^+_{N,1}$. Each cover of $\gamma_\pm$ and $\gamma_{c\pm}$ is a Morse-Bott manifold S^1 since it is isolated, and generates $SH^{S^1,+}(T^*S^3)$. Analyzing this setup, we find:

1. One generator at degree 2, corresponding to the simple retrograde orbit,
2. Two generators at degree $4k + 2$ for $k = 1, \ldots, N - 1$, corresponding to the $(k + 1)$-th cover of the retrograde orbit and k-th cover of the direct orbit.
3. Two generators at degree $4k$ for every $k \in \mathbb{N}$, corresponding to two vertical collision orbits.

Since the degree gap between generators are 2, differentials can be neglected. This result matches the +-part of S^1-equivariant symplectic homology of T^*S^3 described in Proposition 4.4.12 up to degree $4N-2$. By appropriately choosing c, we compute $SH^{S^1,+}(T^*S^3)$ up to the desired degree using only the rotating Kepler orbits.

We assume that the degenerate S^3-families are Morse-Bott. In the planar problem, Morse-Bott-ness of degenerate orbits can be verified using Delauney coordinate, which is described in [Bar65]. To determine the Conley-Zehnder index of degenerate S^3-families, we consider an example illustrated in Figure 4.15. Assume that

$$c_- < c_-^{4,1} < c_+ < c_-^{5,2},$$

so that

$$\mu_{CZ}(\gamma_-^3) = \begin{cases} 14 & \text{if } c = c_-, \\ 18 & \text{if } c = c_+. \end{cases}$$

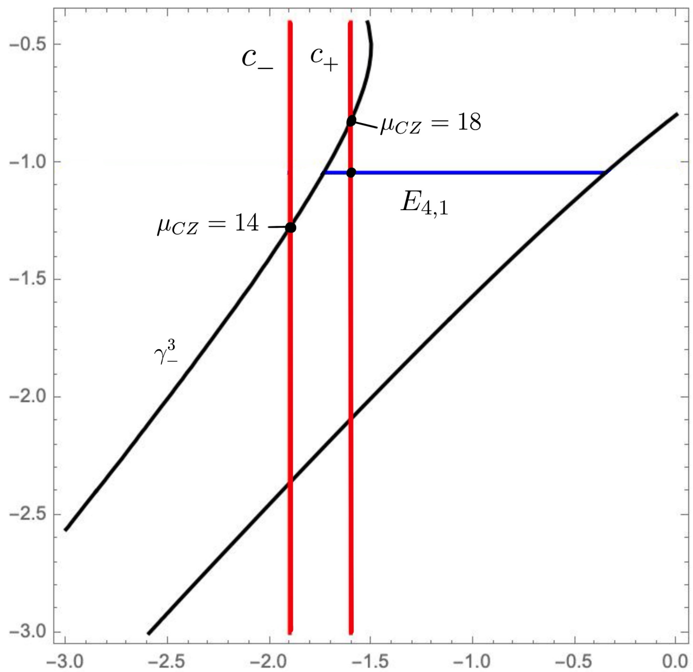

Figure 4.15: Illustration of the local Floer invariance.

There exists an S^3-family of orbits Σ contained in $H^{-1}(c_+)$ that is not contained in $H^{-1}(c_-)$. This family has Kepler energy $E_{3,1}$. From the spectral sequence given in Figure 4.16, the shift of Σ must be 14. It follows that

$$\mu_{CZ}(\Sigma) = sh(\Sigma) + \frac{1}{2}\dim \Sigma/S^1 = 14 + 3/2 = 15.5.$$

We can deduce the same result by considering the fifth cover of the retrograde orbit, which has index 18 at an energy slightly smaller than $c_+^{4,1}$ and index 14 at an energy slightly larger than $c_-^{4,1}$.

To deepen our understanding of the phenomenon, we consider the situation illustrated in Figure 4.17. The two blue lines correspond to $E_{11,1}$ (with a degree shift 42) and $E_{12,2}$ (with a degree shift 46). At $c_- < c_+^{11,1}$, the index of the 10-th cover of the direct orbit, γ_-^{10}, is 50. The contribution of γ_-^{10}, together with the S^3-families of Kepler energy $E_{11,1}$ and $E_{12,2}$ to

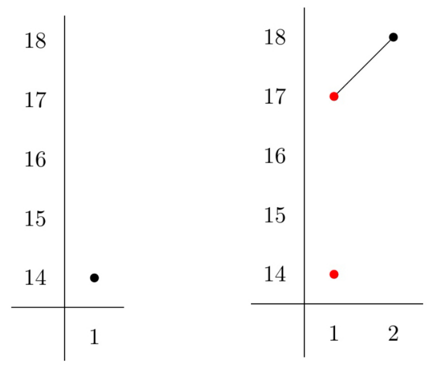

Figure 4.16: Illustration of local Floer invariance for the triple cover of the direct orbit near $E_{4,1}$.

$SH^{S^1,+}(T^*S^3)$ is at the degree 42. Meanwhile, the contribution of the 12-th cover of the retrograde orbit is at the degree 46.

At $c_+ > c_+^{11,1}$, the index of γ_+^{12} becomes 42. In this case, the contribution of γ_+^{12} is the degree 42. Meanwhile, as the $E_{11,1}$-family is no longer present in $H^{-1}(c_+)$, the contribution of γ_-^{10} together with the $E_{12,2}$-family shifts to degree 46. The contributions of the two orbits are effectively interchanged due to the disappearance of the $E_{11,1}$-family, which merges into γ_+^{12}.

In general, we can state the following.

Theorem 4.4.13. *Assume that the degenerate S^3-family of the rotating Kepler problem with Kepler energy $E_{k,l}$ is Morse-Bott. Then the Conley-Zehnder index of the S^3-family of orbits is equal to $4k - 1/2$.*

Proof. From the above discussion, the degree shift of such a family must be equal to $4k-2$. Let the corresponding Morse-Bott submanifold be Σ. Since any bundle over S^3 is trivial, Σ is diffeomorphic to $S^3 \times S^1$. Thus, we have

$$4k - 2 = sh(\Sigma) = \mu_{CZ}(\Sigma) - \frac{1}{2}\dim \Sigma/S^1 = \mu_{CZ}(\Sigma) - 3/2.$$

□

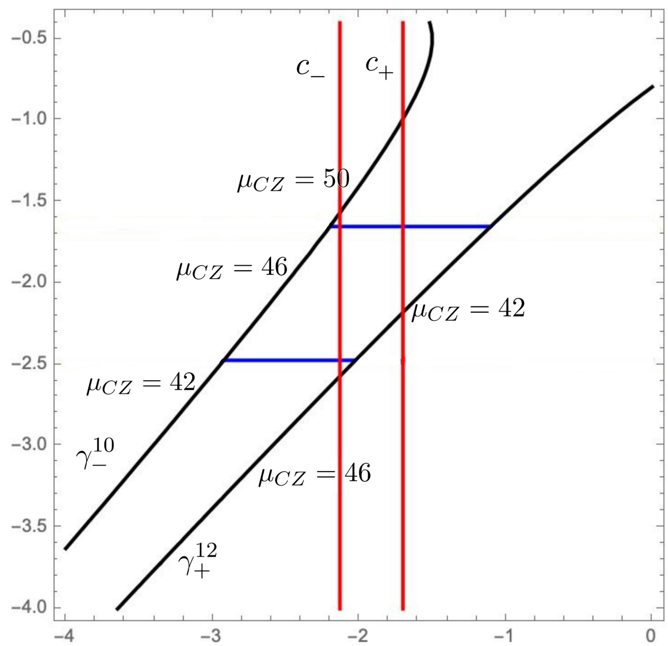

Figure 4.17: Illustration of local Floer invariance for γ_-^{10} and γ_+^{12} near $E_{11,1}$.

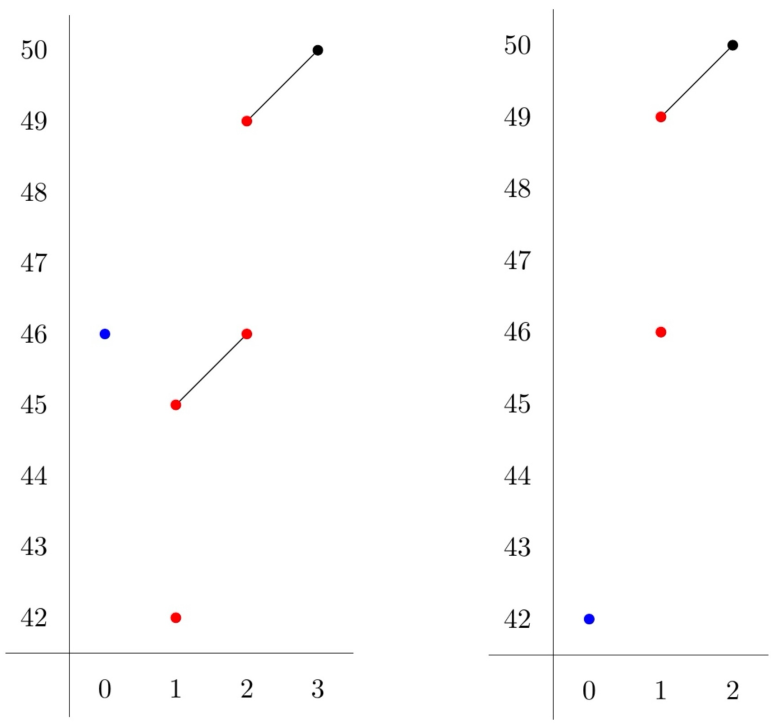

Figure 4.18: Morse-Bott spectral sequence at the energy level $c_- < E_{11,1}$ (left) and $c_+ > E_{11,1}$ (right). Black, blue and red dots correspond to γ_-^{10}, γ_+^{12} and $E_{11,1}$, $E_{12,2}$-families, respectively.

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국문초록

이 논문의 전반부는 볼록성을 가지는 역학적 해밀토니안 벡터장과 볼록한 초곡면 위의 측지선 벡터장에 대한 대역적 절단 초곡면의 존재성에 대해 다룬다. 이는 버코프 원기둥의 일반화로 볼 수 있으며, 케플러 문제를 비롯한 여러 가지 예시를 제시할 것이다.

이 논문의 후반부는 3차원에서 정의된 회전 케플러 문제, 특히 주기성을 가지는 궤도에 대해 다룬다. 먼저 각운동량과 라플라스-렌츠-룽게 벡터를 이용하여 궤도들의 모듈라이 공간을 묘사하는 법에 대해 서술할 것이다. 그 뒤 주기성을 가지는 궤도를 모두 분류하고, 해당 궤도들의 콘리-젠더 인덱스를 계산한 뒤 이를 심플렉틱 호몰로지를 이용하여 해석할 것이다.

주요어휘: 대역적 절단 초곡면, 케플러 문제, 사교기하학, 해밀토니안 동역학, 콘리-젠더 인덱스

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